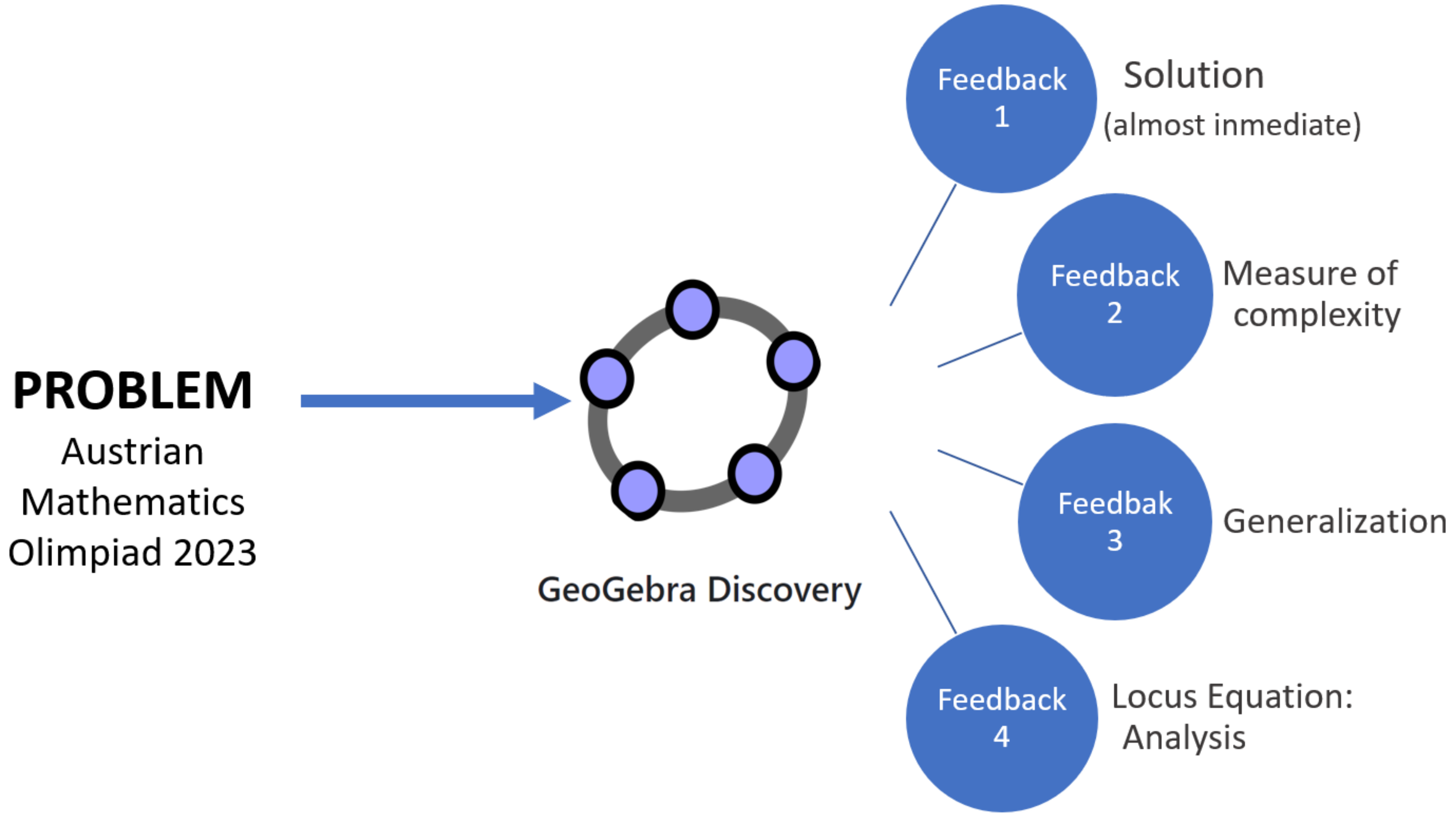
Solving with GeoGebra Discovery an Austrian Mathematics Olympiad problem: lessons learned

Zoltán Kóvacs, Tomás Recio, Belén Ariño, Piedad Tolmos

General Introduction

ADG 2023


PROBLEM
Austrian Mathematics Olimpiad 2023
GeoGebra Discovery
Feedback 1
Solution
(almost inmediate)
Feedback 2
Measure of complexity
Feedbak 3
Generalization
Feedback 4
Locus Equation: Analysis

GeoGebra

GeoGebra Discovery

GeoGebra Discovery

(main developer Zoltán Kovács PPH D Linz)

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https://github.com/kovzol/geogebra/releases Versions GeoGebra 5 Discovery and GeoGebra 6 Discovery off-line

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http://autgeo.online GeoGebra 6 Discovery on-line

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http://autgeo.online/ag/automated-geometer.html?offline=1 Automated Geometer

Sei ABCD eine Raute mit 4BAD < 90'. Der Kreis durch D mit Mitteipunkt A schneide die Gerade CD ebzweites Mal im Punkt,E. Der Schnittpunkt der Geradet BE und AC sei S.

Man beweise, dass die Punkte A, S, D und E auf einem Kreis liegen

54. Österreichische Mathematik-Olympiade

Regionalwettbewerb für Fortgeschrittene

30. März 2023

1. Es seien a, b und c reelle Zahlen mit $0 \leq a, b, c \leq 2$. Man beweise, dass
$$(a-b)(b-c)(a-c) \leq 2$$
gilt, und man gebe an, wann Gleichheit eintritt.

(Karl Czakler)

2. Sei $ABCD$ eine Raute mit $\sphericalangle BAD < 90°$. Der Kreis durch D mit Mittelpunkt A schneide die Gerade CD ein zweites Mal im Punkt E. Der Schnittpunkt der Geraden BE und AC sei S.

   Man beweise, dass die Punkte A, S, D und E auf einem Kreis liegen.

(Karl Czakler)

3. Man bestimme alle natürlichen Zahlen $n \geq 2$, für die es zwei Anordnungen $(a_1, a_2, \ldots, a_n)$ und $(b_1, b_2, \ldots, b_n)$ der Zahlen $1, 2, \ldots, n$ gibt, sodass $(a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$ aufeinander folgende natürliche Zahlen sind.

(Walther Janous)

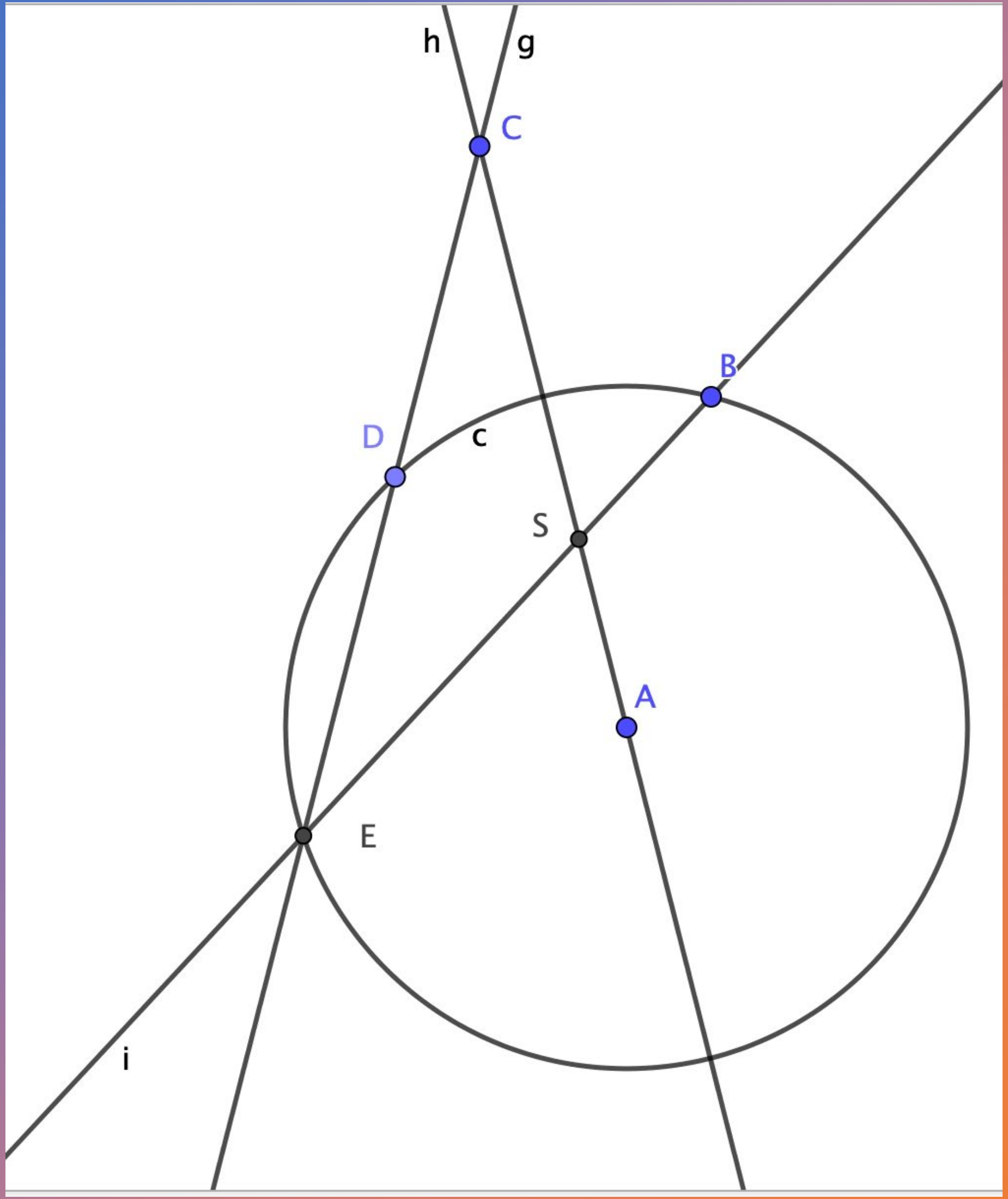
4. Man bestimme alle Paare (x, y) von positiven ganzen Zahlen, sodass für $d = \mathrm{ggT}(x, y)$ die Gleichung
$$xyd = x + y + d^2$$
gilt.

(Walther Janous)

Arbeitszeit: 4 Stunden.
Bei jeder Aufgabe können 8 Punkte erreicht werden.

Let ABCD be a rhombus with angle BAD < 90◦. The circle through D with center A intersects straight line CD a second time at point E. The intersection of the lines BE and AC is S.

Prove that the points A, S, D and E lie on a circle.



First, we have chosen some free points A,B, then the circle c centered at A through B, then another point D on this circle, such that angle BAD < 90°. Next, we have built the (hidden) segment f = BD, and point C as the symmetrical of A with respect to f. Thus, ABCD is a rhombus. Finally, points E,S are displayed, following the hypotheses, as the intersection of line CD and c (ditto, as the intersection of line BE and AC).

A = (0.7, -0.84)

B = (1.56, 2.52)

c: $(x - 0.7)^2 + (y + 0.84)^2 = 12.03$

D = (-1.66, 1.71)

f = 3.32

C = (-0.8, 5.07)

g: -3.36x + 0.86y = 7.03

E = (-2.59, -1.94)

h: 5.91x + 1.5y = 2.88

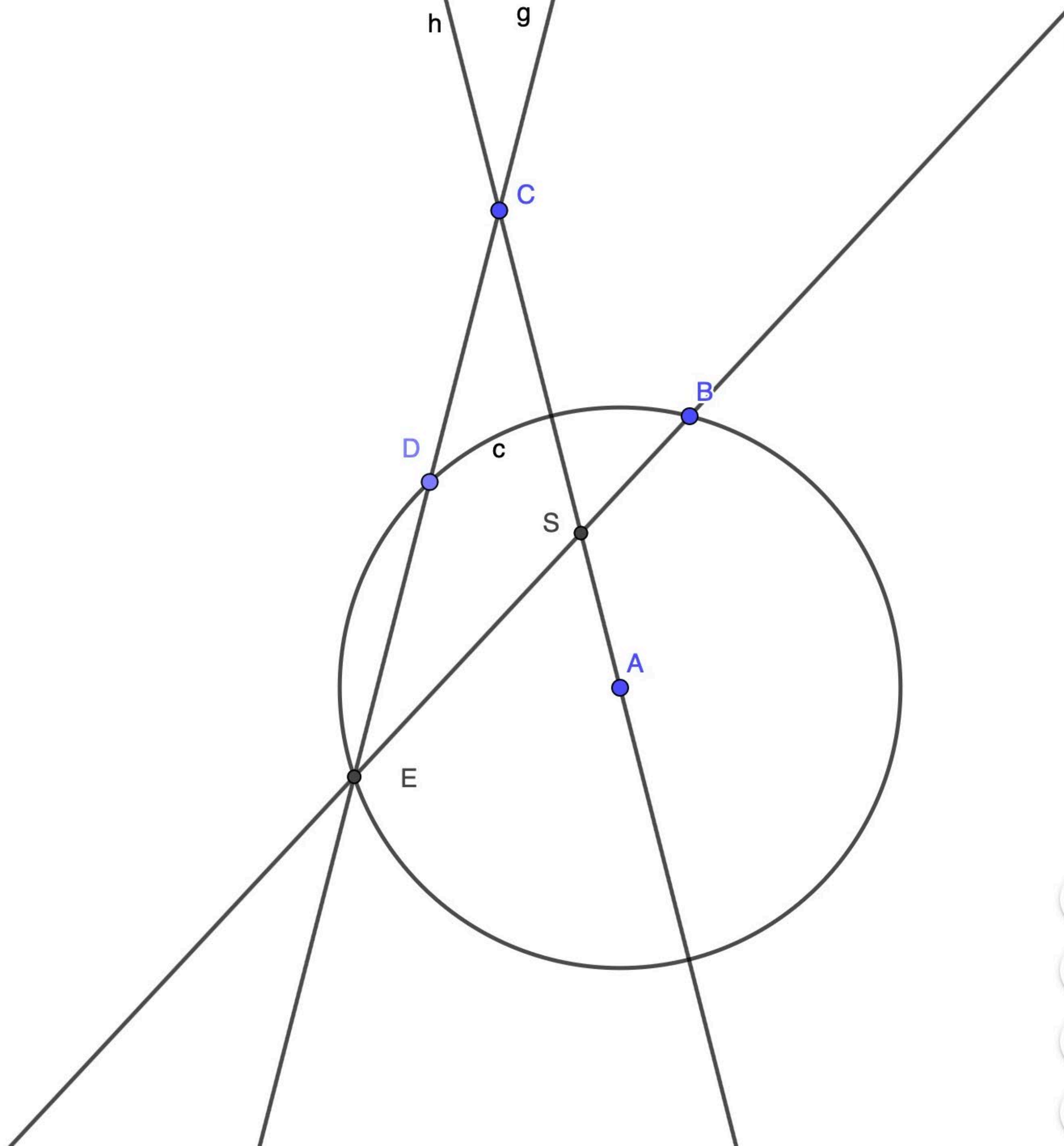
i: -4.46x + 4.15y = 3.49

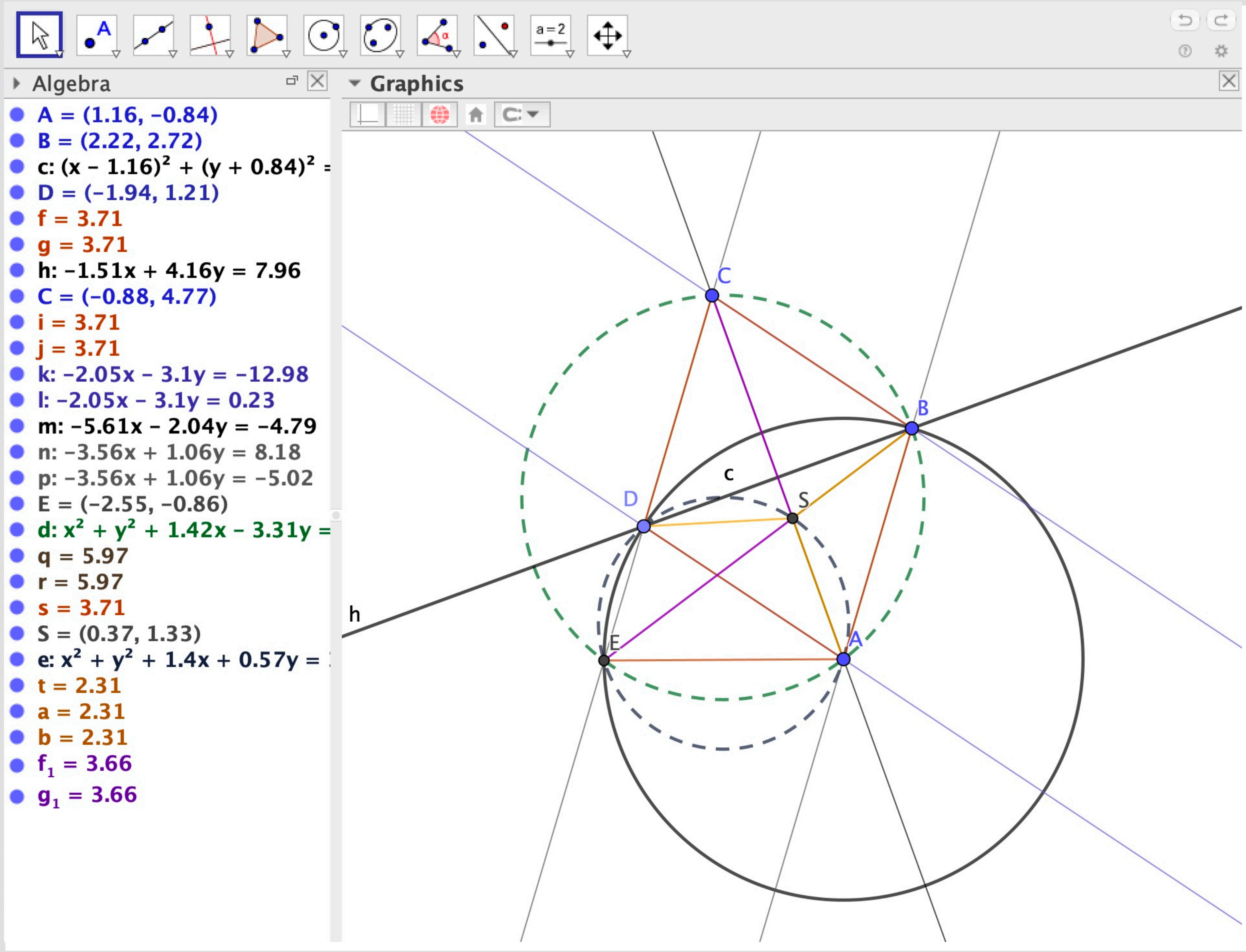
S = (0.22, 1.07)

a = Prove(AreConcyclic(A, S, D, E))

b = {true, {"…"}}

Input…




a = 2
Algebra
Graphics
A = (1.16, -0.84)
B = (2.22, 2.72)
c: (x - 1.16)² + (y + 0.84)² =
D = (-1.94, 1.21)
f = 3.71
g = 3.71
h: -1.51x + 4.16y = 7.96
C = (-0.88, 4.77)
i = 3.71
j = 3.71
k: -2.05x - 3.1y = -12.98
l: -2.05x - 3.1y = 0.23
m: -5.61x - 2.04y = -4.79
n: -3.56x + 1.06y = 8.18
p: -3.56x + 1.06y = -5.02
E = (-2.55, -0.86)
d: x² + y² + 1.42x - 3.31y =
q = 5.97
r = 5.97
s = 3.71
S = (0.37, 1.33)
e: x² + y² + 1.4x + 0.57y =
t = 2.31
a = 2.31
b = 2.31
f1 = 3.66
g1 = 3.66
C
B
c
D
S
h
E
A

54. Österreichische Mathematik-Olympiade

Regionalwettbewerb für Fortgeschrittene — Oberösterreich/Salzburg/Tirol/Vorarlberg

	1	2	3	4
0	36	19	32	9
1	12	10	19	26
2	13	6	6	9
3	5	3	0	14
4	4	3	1	5
5	0	4	3	3
6	0	1	2	2
7	0	4	0	2
8	5	25	12	5
Durchschnitt	**1,45**	**3,96**	**2,11**	**2,49**

A = (-4.37, 4.31)

B = (-1.69, 6.97)

c: $(x + 4.37)^2 + (y - 4.31)^2 = 14.26$

D = (-5.99, 7.72)

f: $0.75x + 4.3y = 28.7$

C = (-4.11, 9.71)

g: $1.99x - 1.88y = -26.43$

E = (-7.87, 5.74)

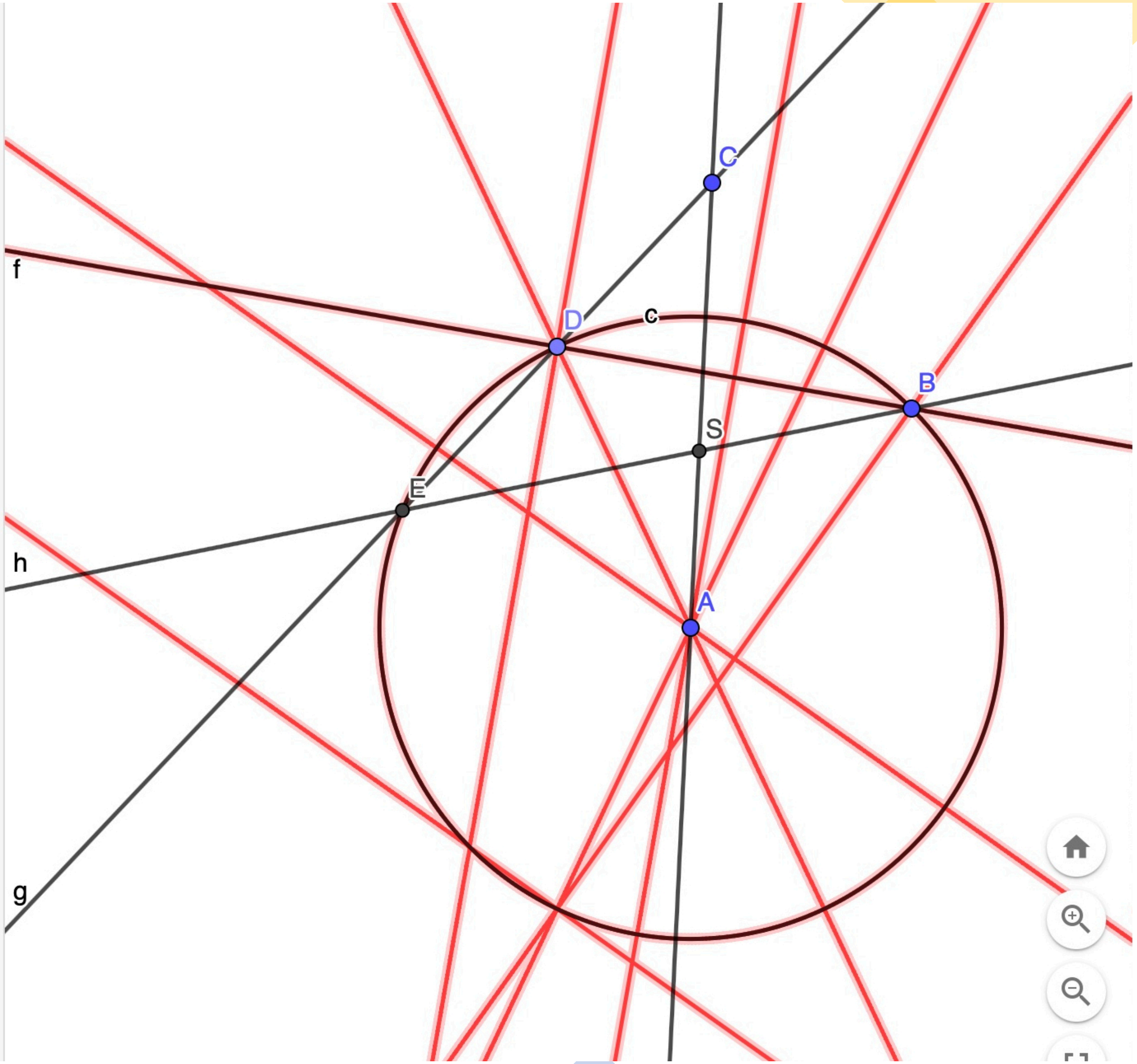
h: $1.23x - 6.18y = -45.14$

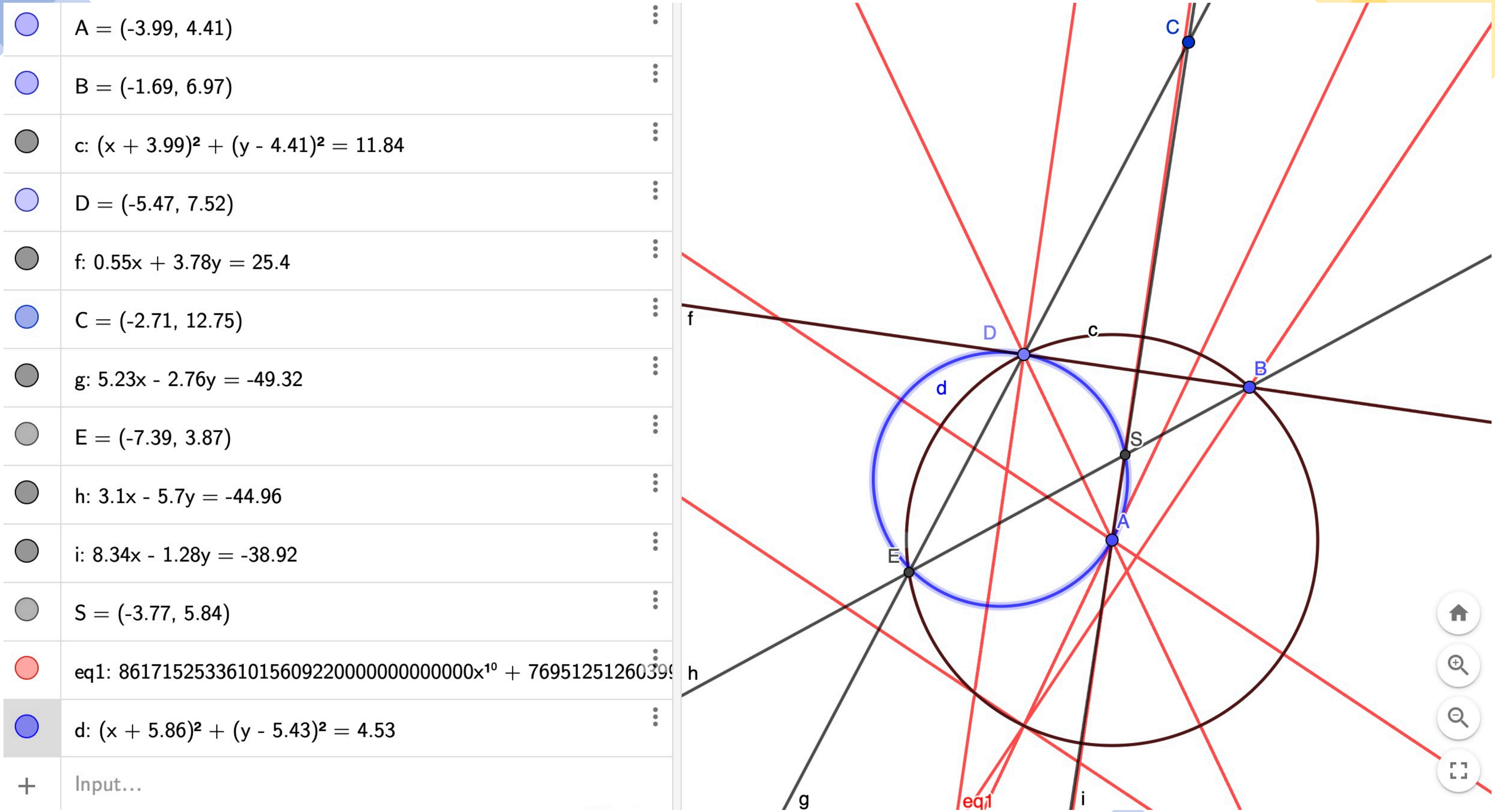
i: $5.4x - 0.26y = -24.72$

S = (-4.27, 6.45)

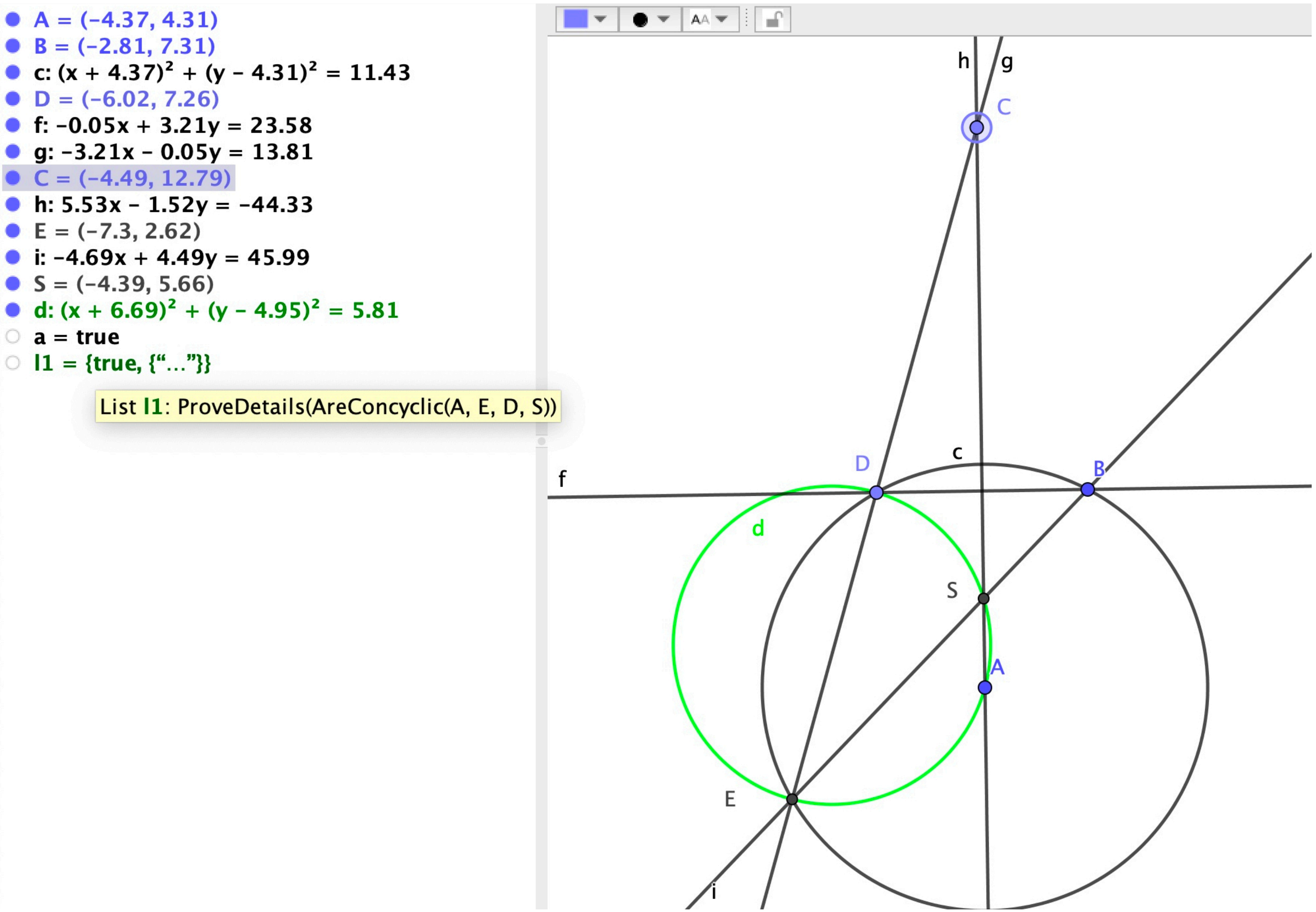
eq1 : LocusEquation(AreConcyclic(D, E, A, S), C)

+ Input...

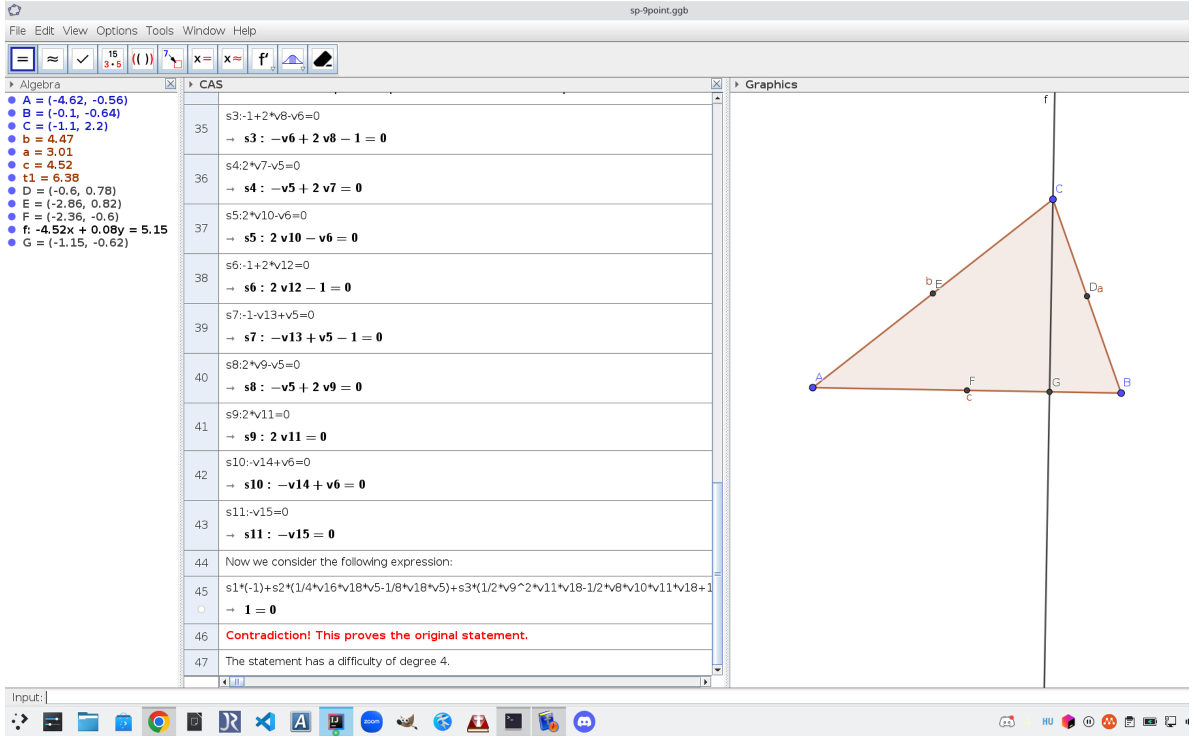


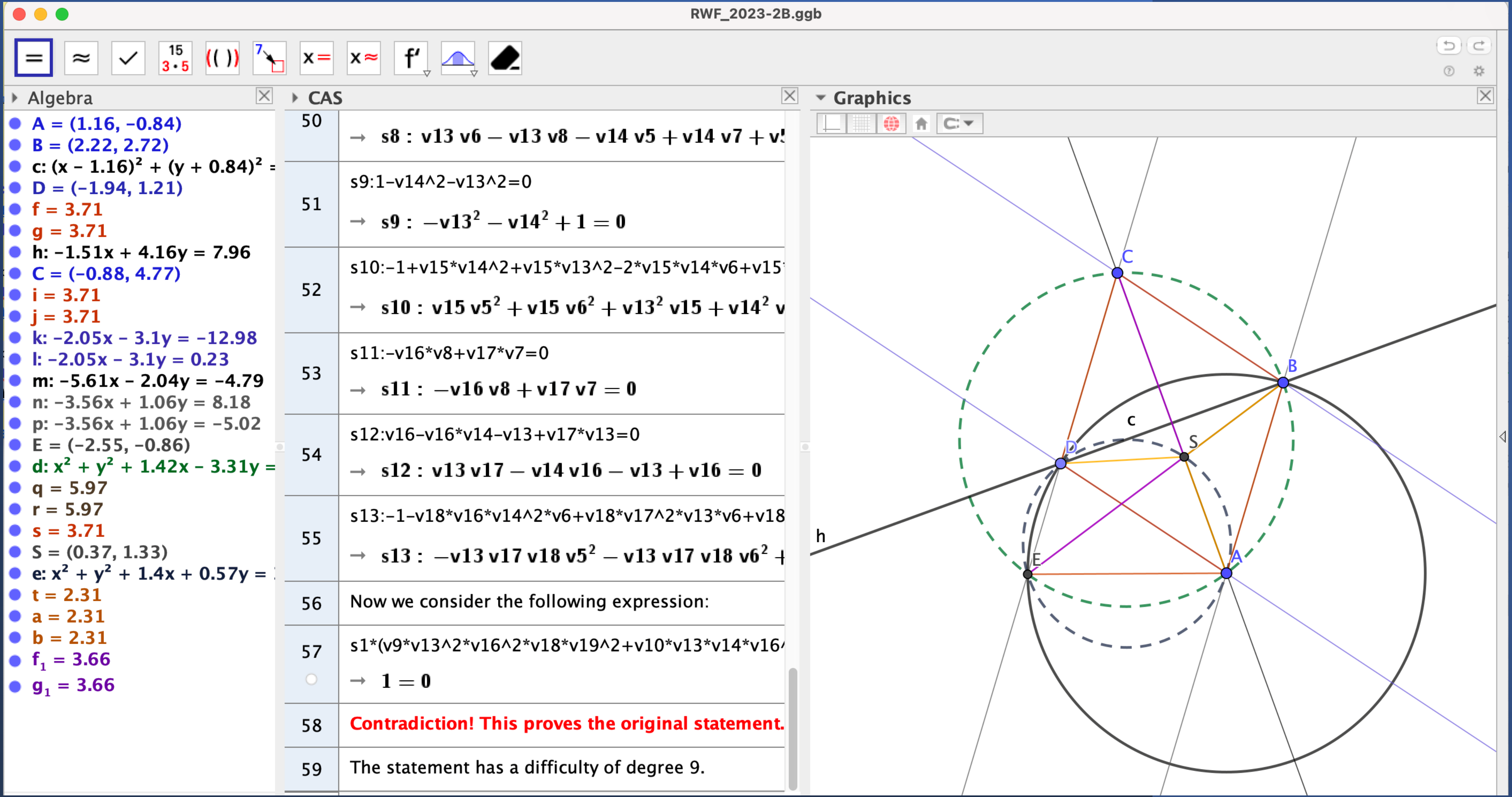


A = (-3.99, 4.41)
B = (-1.69, 6.97)
c: (x + 3.99)² + (y - 4.41)² = 11.84
D = (-5.47, 7.52)
f: 0.55x + 3.78y = 25.4
C = (-2.71, 12.75)
g: 5.23x - 2.76y = -49.32
E = (-7.39, 3.87)
h: 3.1x - 5.7y = -44.96
i: 8.34x - 1.28y = -38.92
S = (-3.77, 5.84)
eq1: 8617152533610156092200000000000000x¹⁰ + 769512512603
d: (x + 5.86)² + (y - 5.43)² = 4.53
Input...
C
f
D
c
B
d
S
A
E
h
g
eq1
i


A = (−4.37, 4.31)
B = (−2.81, 7.31)
c: (x + 4.37)² + (y − 4.31)² = 11.43
D = (−6.02, 7.26)
f: −0.05x + 3.21y = 23.58
g: −3.21x − 0.05y = 13.81
C = (−4.49, 12.79)
h: 5.53x − 1.52y = −44.33
E = (−7.3, 2.62)
i: −4.69x + 4.49y = 45.99
S = (−4.39, 5.66)
d: (x + 6.69)² + (y − 4.95)² = 5.81
a = true
l1 = {true, {"..."}}
List l1: ProveDetails(AreConcyclic(A, E, D, S))
h
g
C
D
c
B
f
d
S
A
E
i

Grading problem 2




RWF_2023-2B.ggb
Algebra
A = (1.16, −0.84)
B = (2.22, 2.72)
c: (x − 1.16)² + (y + 0.84)² =
D = (−1.94, 1.21)
f = 3.71
g = 3.71
h: −1.51x + 4.16y = 7.96
C = (−0.88, 4.77)
i = 3.71
j = 3.71
k: −2.05x − 3.1y = −12.98
l: −2.05x − 3.1y = 0.23
m: −5.61x − 2.04y = −4.79
n: −3.56x + 1.06y = 8.18
p: −3.56x + 1.06y = −5.02
E = (−2.55, −0.86)
d: x² + y² + 1.42x − 3.31y =
q = 5.97
r = 5.97
s = 3.71
S = (0.37, 1.33)
e: x² + y² + 1.4x + 0.57y =
t = 2.31
a = 2.31
b = 2.31
f₁ = 3.66
g₁ = 3.66
CAS
50 → s8 : v13 v6 − v13 v8 − v14 v5 + v14 v7 + v
51 s9:1−v14^2−v13^2=0
→ s9 : −v13² − v14² + 1 = 0
52 s10:−1+v15*v14^2+v15*v13^2−2*v15*v14*v6+v15
→ s10 : v15 v5² + v15 v6² + v13² v15 + v14² v
53 s11:−v16*v8+v17*v7=0
→ s11 : −v16 v8 + v17 v7 = 0
54 s12:v16−v16*v14−v13+v17*v13=0
→ s12 : v13 v17 − v14 v16 − v13 + v16 = 0
55 s13:−1−v18*v16*v14^2*v6+v18*v17^2*v13*v6+v18
→ s13 : −v13 v17 v18 v5² − v13 v17 v18 v6² +
56 Now we consider the following expression:
57 s1*(v9*v13^2*v16^2*v18*v19^2+v10*v13*v14*v16
→ 1 = 0
58 Contradiction! This proves the original statement.
59 The statement has a difficulty of degree 9.
Graphics
C
B
c
S
D
h
E
A

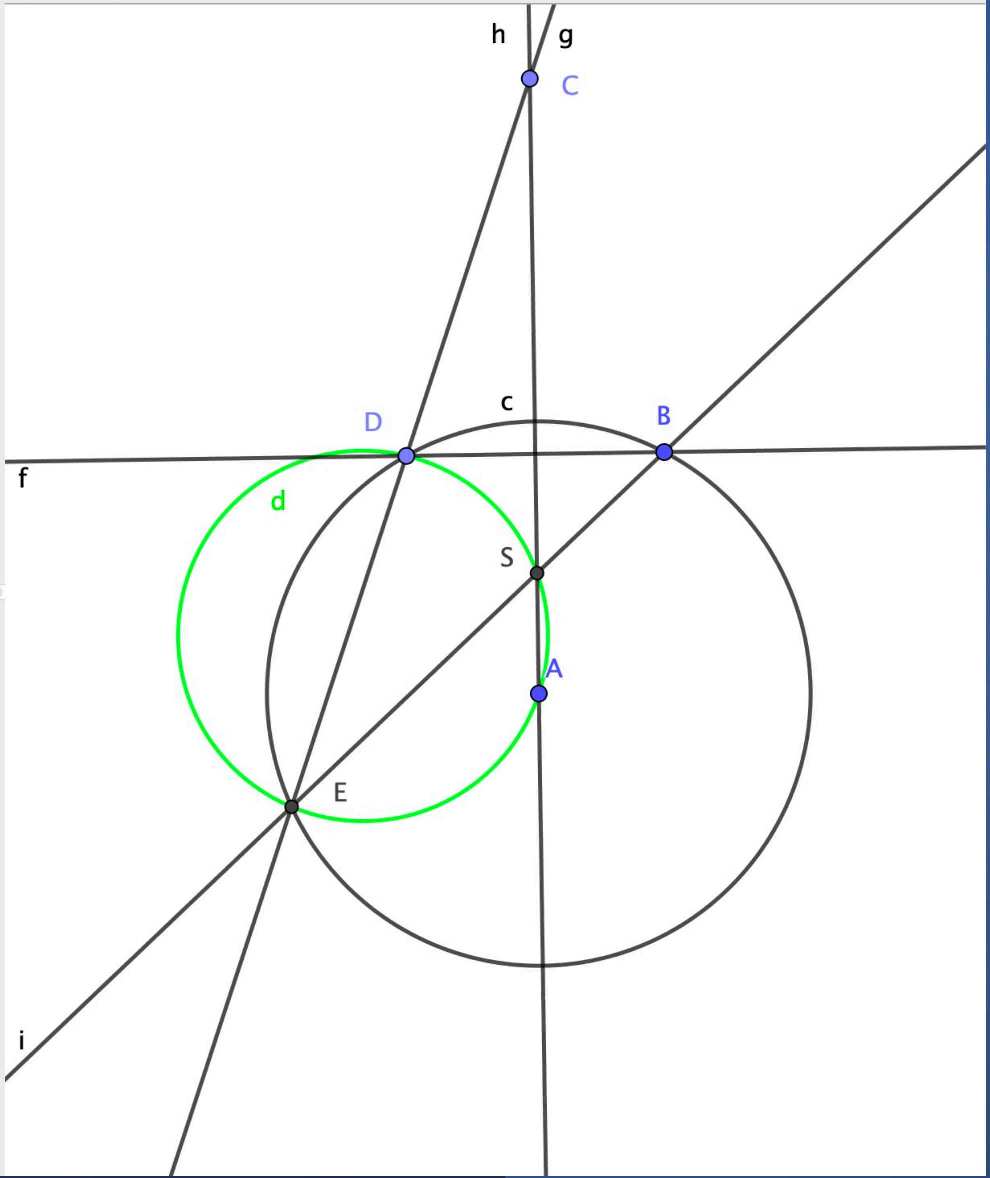
Algebra

- A = (−4.37, 4.31)
- B = (−2.81, 7.31)
- c: $(x + 4.37)^2 + (y - 4.31)^2 =$
- D = (−6.02, 7.26)
- f: −0.05x + 3.21y = 23.58
- g: −3.21x − 0.05y = 13.81
- C = (−4.48, 11.95)
- h: 4.69x − 1.54y = −39.36
- E = (−7.45, 2.9)
- i: −4.41x + 4.64y = 46.26
- S = (−4.39, 5.81)
- d: $(x + 6.56)^2 + (y - 5.03)^2 =$

CAS

44	s4:v9*v8−v10*v7=0 → $s4 : -v10\ v7 + v8\ v9 = 0$
45	s5:v11*v10−v12*v9−v11*v6+v9*v6+v12*v5−v10*v5= → $s5 : v10\ v11 - v10\ v5 - v11\ v6 + v12\ v5 -$ v
46	s6:−v12^2−v11^2+v4^2+v3^2=0 → $s6 : -v11^2 - v12^2 + v3^2 + v4^2 = 0$
47	s7:−1+v13*v12^2+v13*v11^2−2*v13*v12*v6+v13*v → $s7 : v13\ v5^2 + v13\ v6^2 + v11^2\ v13 + v12^2\ v1$
48	s8:v14*v8−v15*v7=0 → $s8 : v14\ v8 - v15\ v7 = 0$
49	s9:v14*v12−v15*v11−v14*v4+v11*v4+v15*v3−v12*v → $s9 : -v11\ v15 + v11\ v4 + v12\ v14 - v12\ v3$
50	s10:−1+v16*v14*v12^2*v6−v16*v15^2*v11*v6−v16* → $s10 : v11\ v15\ v16\ v5^2 + v11\ v15\ v16\ v6^2 -$ v
51	Now we consider the following expression:
52	s1*(2*v7^2*v13*v14^2*v16*v17*v3*v4−4*v7*v10*v1 → $1 = 0$
53	**Contradiction! This proves the original statement.**
54	The statement has a difficulty of degree 10.

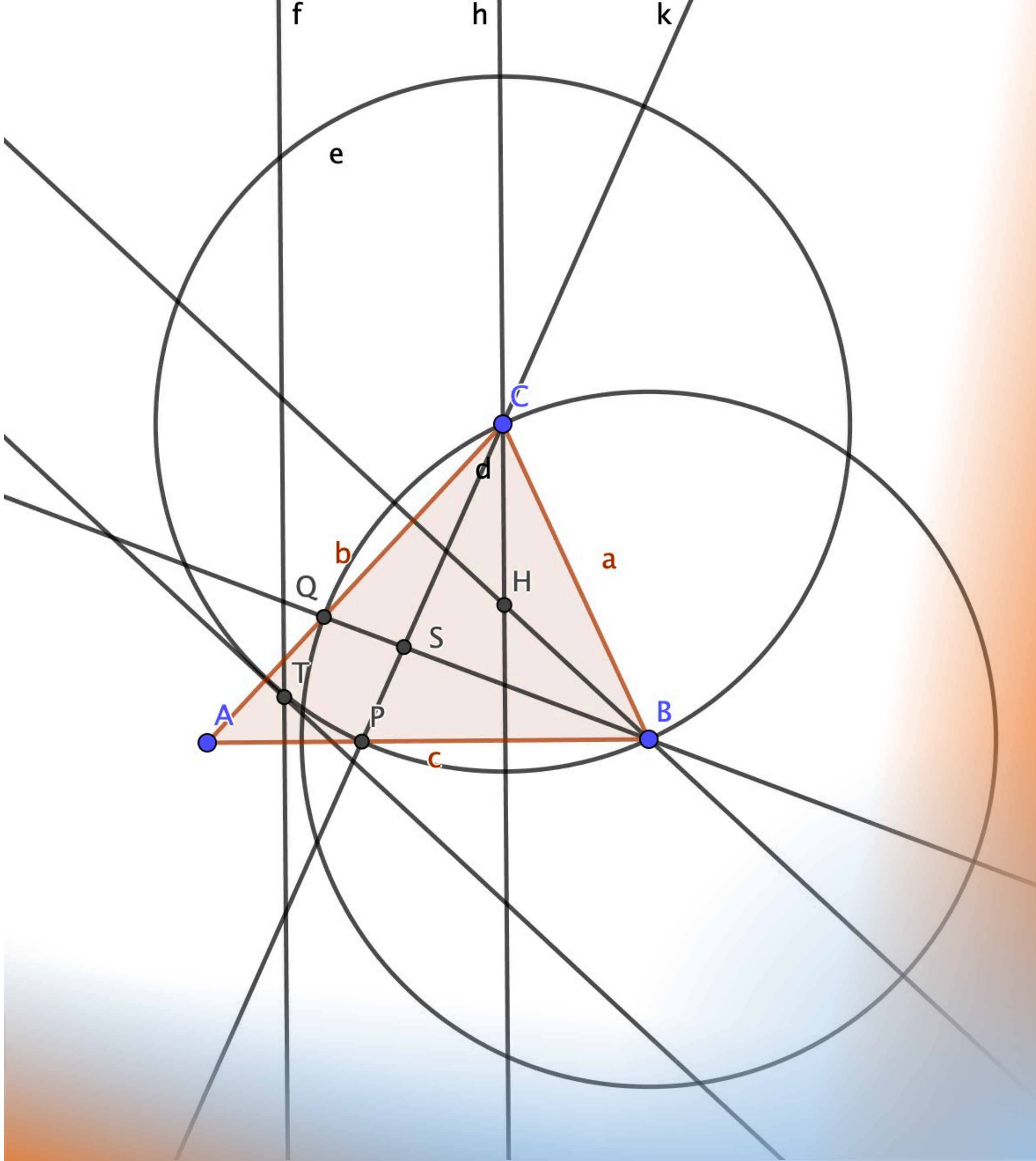
Graphics



Other olympic problems and GG Discovery

EGMO 2022

Problem 1. Let ABC be an acute triangle with BC < AB and BC < AC. Consider the points P and Q on the segments AB and AC, respectively, such that P ≠ B, Q ≠ C and BQ = BC = CP. Let T be the circumcenter of triangle APQ, H be the orthocenter of triangle ABC, and S be the point of intersection of lines BQ and CP. Prove that the points T, H and S are on the same line.



- B = (11.81, 0.9)
- C = (6.68, 14.67)
- f: 0.04x + 13.55y = 12.72
- g: −13.73x + 8.42y = 31.77
- $c_2: (x - 11.81)^2 + (y - 0.9)^2 = 21$
- Q = (−2.79, −0.77)
- $d: (x - 6.68)^2 + (y - 14.67)^2 = 21$
- P = (1.46, 0.93)
- h: 13.74x − 5.22y = 15.16
- i: 1.67x − 14.6y = 6.63
- S = (0.97, −0.34)
- j: −10.35x + 0.03y = −68.64
- k: 9.47x + 15.44y = 125.73
- H = (6.64, 4.07)
- l: −3.19x + 0.01y = 0.45
- m: 1.05x + 1.72y = −2.24
- T = (−0.15, −1.21)
- b = 16.1
- a = 14.69
- c = 13.55
- t1 = 93.14
- n: −4.41x + 5.67y = −6.23
- $e: x^2 + y^2 + 2.21x - 2.36y = 4.03$
- $p: x^2 + y^2 - 13.25x + 3.52y = -13.04$
- $q: x^2 + y^2 + 4.1x - 18.79y = 11.49$
- r: −3.17x − 5.17y = −42.05
- s: 10.6x − 0.04y = 70.31
- t: −13.77x − 5.13y = −167.24
- f_1: −3.12x + 8.38y = 13.34
- o = true
- l1 = {true, {“AreCongruent[a,c]”, “AreEqual[A,B]”}}

Relation

It is generally true that:

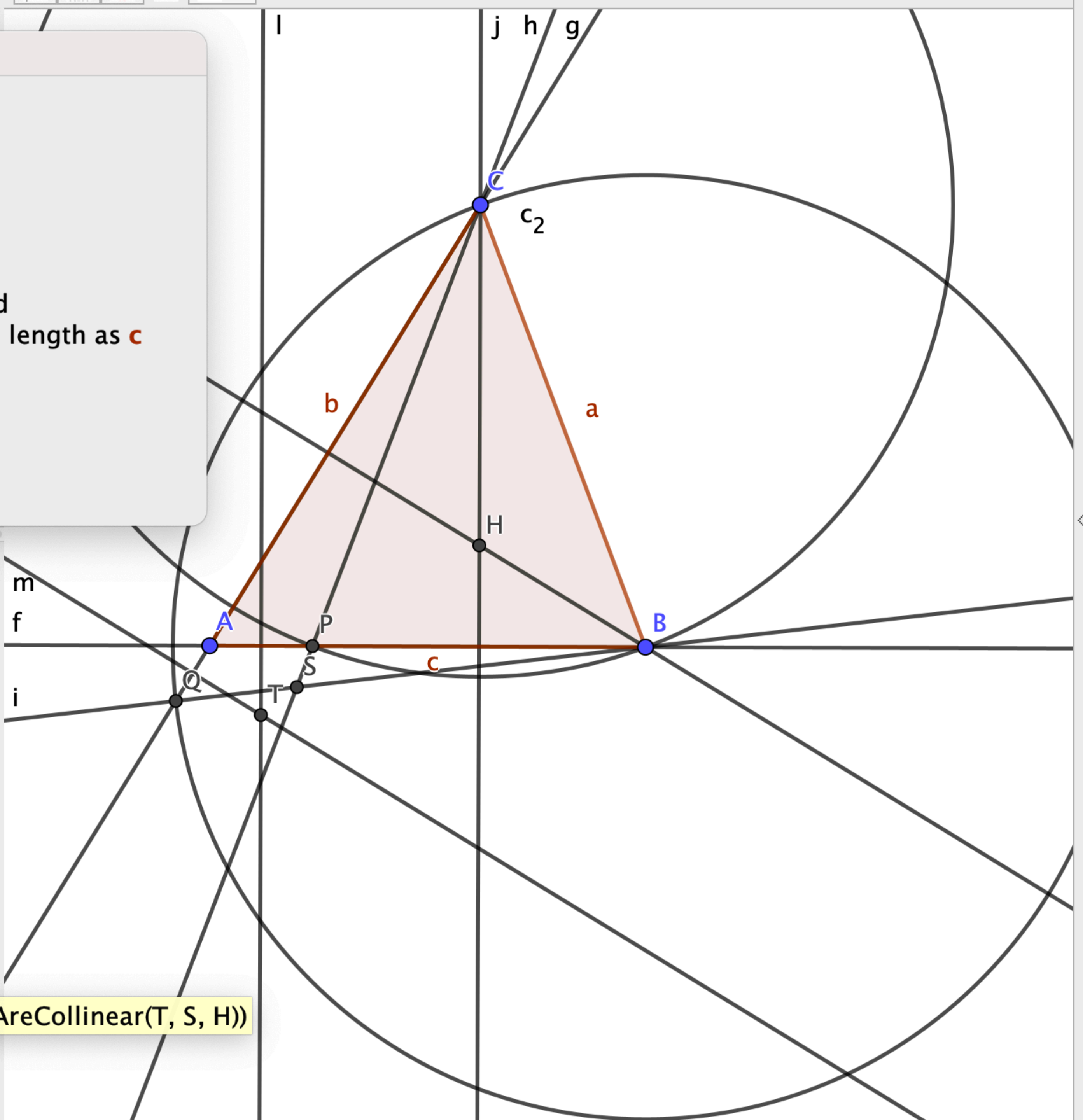
- T, S and H are collinear

under the condition:

- A and B are not equal and
- a does not have the same length as c

OK

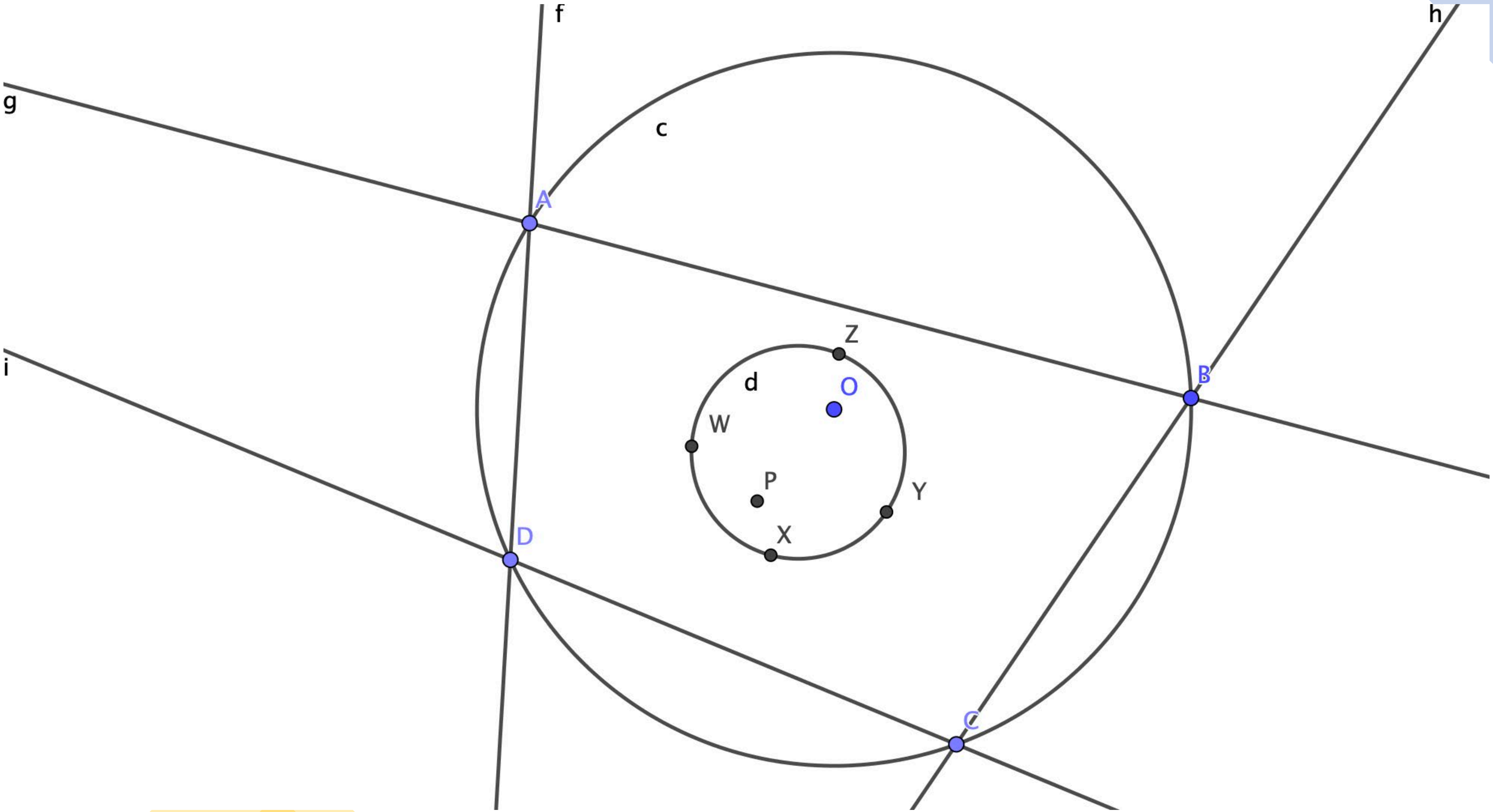
List l1: ProveDetails(AreCollinear(T, S, H))



EGMO 2022

Problem 6. Let ABCD be a cyclic quadrilateral with circumcenter O. Let X be the point of intersection of the bisectors of the angles ∠DAB and ∠ABC; point Y that of the bisectors of the angles ∠ABC and ∠BCD; Z is the bisector of the angles ∠BCD and ∠CDA; and let W be the point of intersection of the bisectors of the angles ∠CDA and ∠DAB. Let P be the point of intersection of lines AC and BD. Points O, P, X, Y, Z, and W are assumed to be distinct.

Prove that O, X, Y, Z, and W lie on the same circle if and only if P, X, Y, Z, and W lie on the same circle.


f
h
g
c
A
Z
i
d
O
B
W
P
Y
X
D
C

EGMO 2022

SPANISH TEAM SCORE (7 pt. max)

- Problem 1: C1=1, C2=7, C3=0, C4=1
- Problem 6: C1=0, C2=1, C3=0, C4=0
- Note that C1 and C3 obtained Mention and C2 Bronze Medal

IN GENERAL

- Mean score for Problem 1 in all the participants was slightly more than 4.6 points, compared to 1.05 for the mean score for Problem 6, the lowest after Problem 3 (0.78).

Final reflection…

We have ilustrated…

1. The ability of GeoGebra Discovery Automated Reasoning Tools (ART) to immediately solve a problem presented at a regional Mathematics Olympiad, that the recent GeoGebra ART complexity measure ranks quite highly,
2. The use of GeoGebra Discovery as a decisive auxiliary tool to develop and confirm new, non-trivial, conjectures, such as the generalization of the pro- posed problem,
3. The need to change the methodological focus when working with locus com- putation in the classroom with Dynamic Geometry programs, from finding equations and displaying its graph, to analyzing and obtaining the geometric characteristics of the involved locus, and its construction, by using GeoGebra Discovery ART.
4. Possibility to compare the behaviour the GGD and human’s concern the dificulty of problems.
5. The opportunity to consider simultaneously all these items around a single problem, is probably the most relevant contribution of this communication.

References to other olympic problems and other authors dealing with automatic reasoning and olympiads.

EGMO in particular. Comparing with human behavior

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1. Ariño-Morera, B.; Recio, T.; Tolmos, P.: Olympic geometry problems: human vs. machine. Communication to the CADGME (Digital Tools in Mathematics Educa- tion) 2022 Conference. Abstracts available at https://drive.google.com/file/d/1qF4ceMg6gNklOPa1JVkgKND1dOqNmyka/viewpp

2. Ariño-Morera, M. B.: GeoGebra Discovery at EGMO 2022. Revista Do Instituto GeoGebra Internacional De São Paulo, 11(2), 005-016.2022. https://doi.org/10.23925/2237-9657.2022.v11i2p005-016

Хвала вам
Thank you!

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