Showing proofs, assessing difficulty with GeoGebra Discovery

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ADG 2023

Belgrade, Serbia

20 September 2023

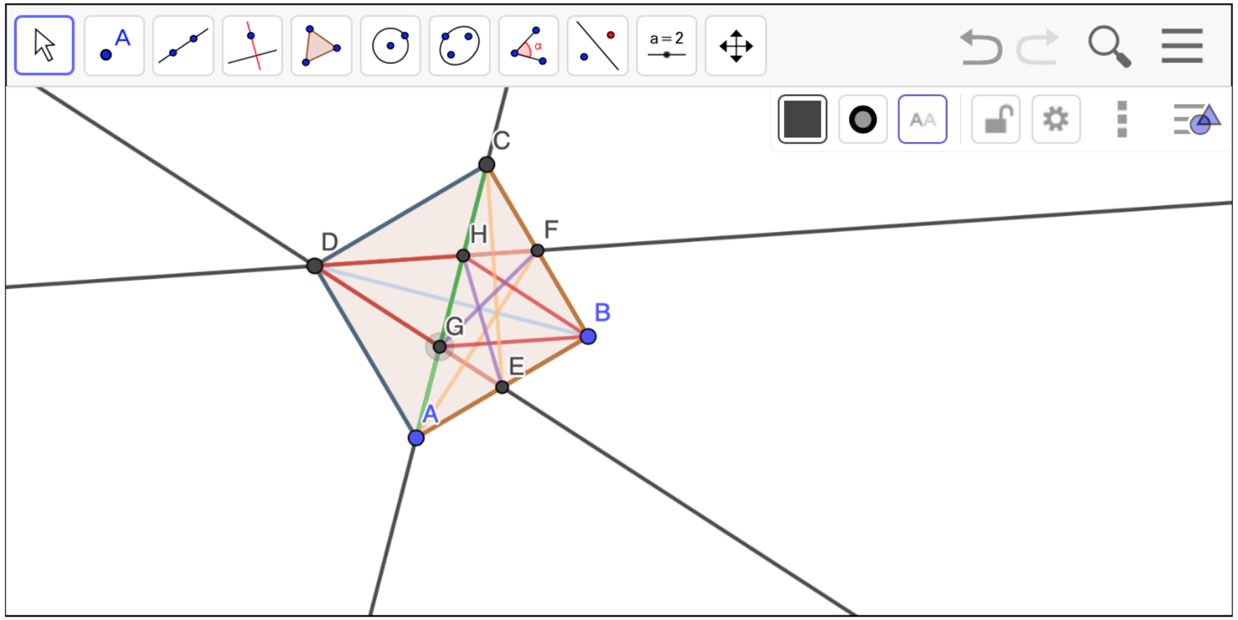
GeoGebra Discovery
(main developer Z. K. + advisory/tester team)

- https://github.com/kovzol/geogebra/releases Versions GeoGebra 5 Discovery and GeoGebra 6 Discovery off-line
- http://www.autgeo.online/geogebra -discovery/ GeoGebra 6 Discovery on-line
- http://www.autgeo.online/ag /automated-geometer.html?offline=1 Automated Geometer

Welcome to the Automated Geometer!

Using GeoGebra 5.0.495.0 (offline).

Let us consider this initial input construction (only the visible points will be observed) :



Select relations to check:

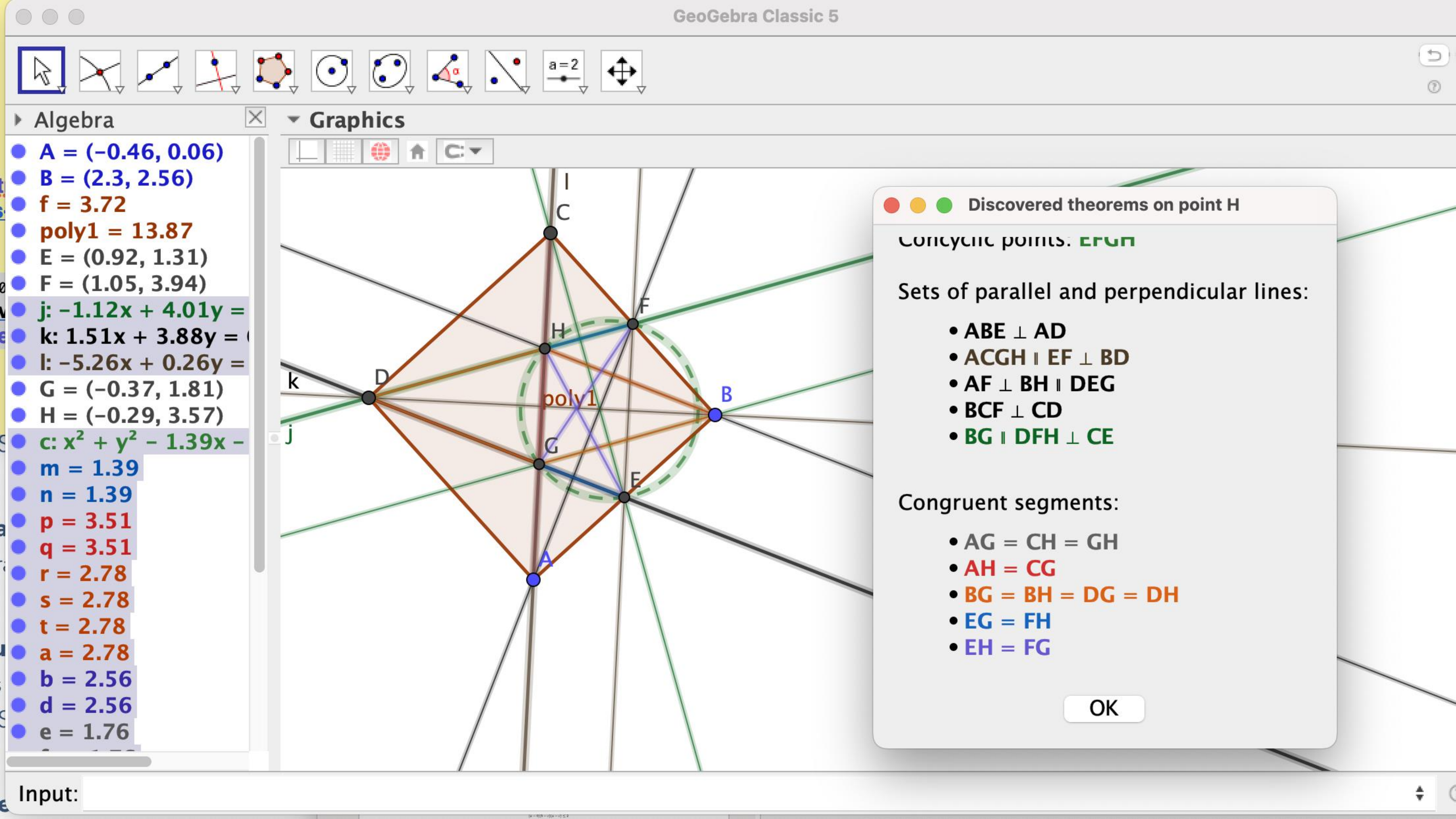
Collinearity of three points
Equality of distances between two points
Perpendicularity of segments defined by two points
Parallelism of segments defined by two points
Concyclicity of four points

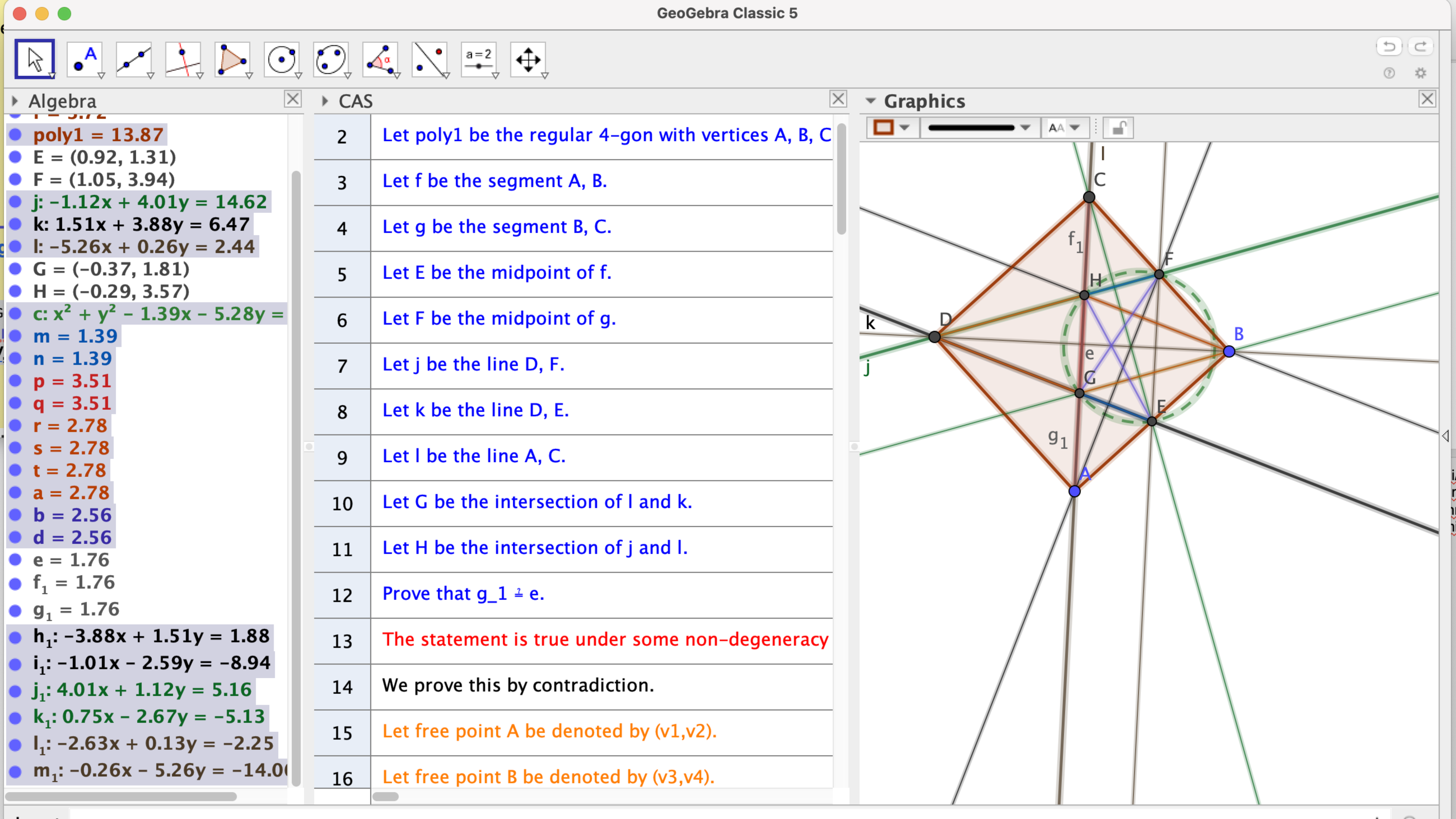
The following theorems can be proven:

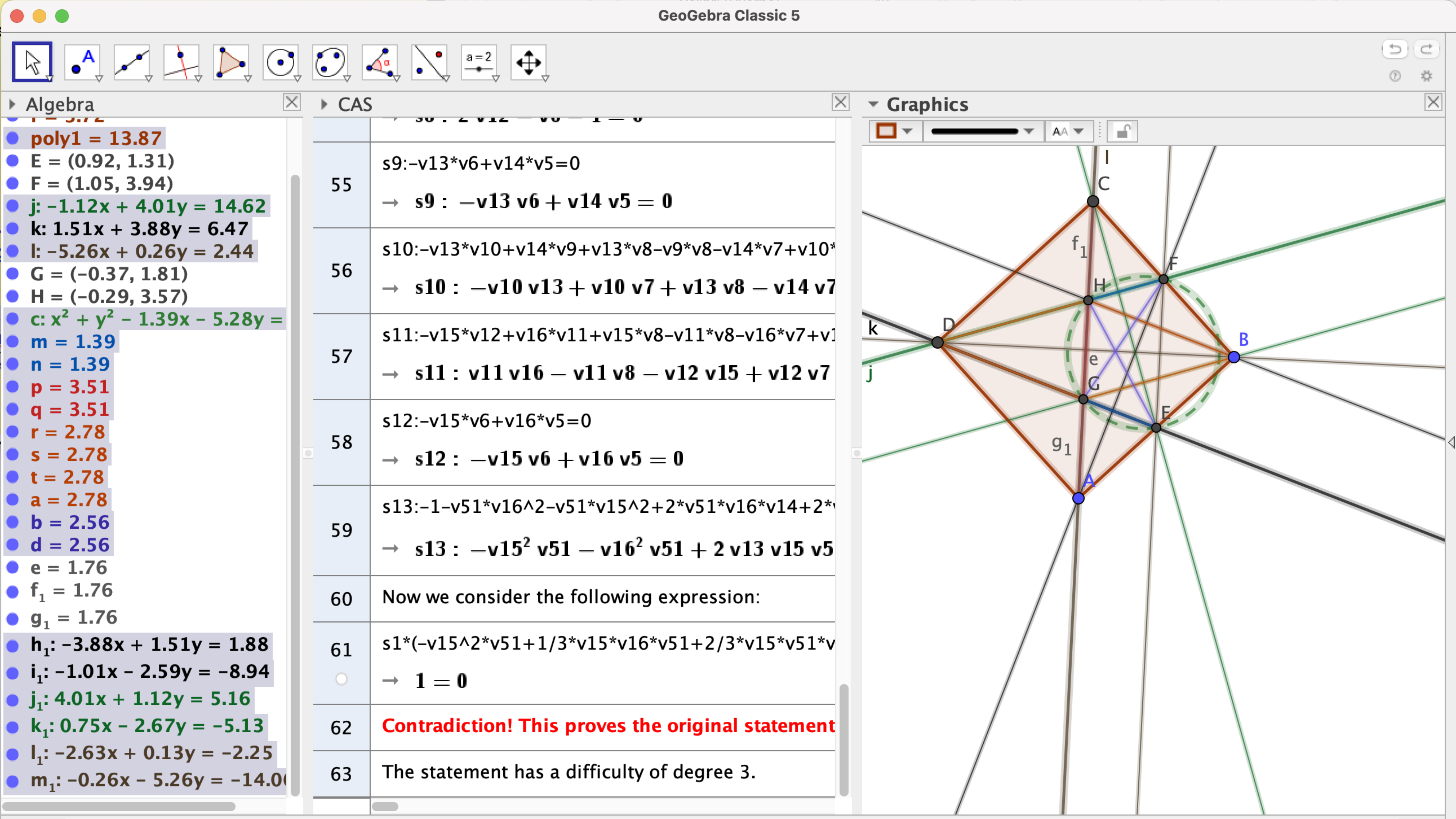
1. AB=AD
2. AB=BC
3. AB=CD
4. AC=BD
5. AD=BC
6. AD=CD
7. AE=BE
8. AE=BF
9. AE=CF
10. AF=CE
11. AF=DE
12. AF=DF
13. AG=CH
14. AG=GH
15. AH=CG
16. BC=CD
17. BE=BF
18. BE=CF
19. BF=CF
20. BG=BH
21. BG=DG
22. BG=DH
23. BH=DG
24. BH=DH
25. CE=DE
26. CE=DF
27. CH=GH
28. DE=DF
29. DG=DH
30. EG=FH
31. EH=FG

Finished, found 31 theorems among 378 possible statements.
Elapsed time: 0h 0m 2s

Restart with a new or the same experiment


GeoGebra Classic 5
Algebra
A = (−0.46, 0.06)
B = (2.3, 2.56)
f = 3.72
poly1 = 13.87
E = (0.92, 1.31)
F = (1.05, 3.94)
j: −1.12x + 4.01y =
k: 1.51x + 3.88y =
l: −5.26x + 0.26y =
G = (−0.37, 1.81)
H = (−0.29, 3.57)
c: x² + y² − 1.39x −
m = 1.39
n = 1.39
p = 3.51
q = 3.51
r = 2.78
s = 2.78
t = 2.78
a = 2.78
b = 2.56
d = 2.56
e = 1.76
Graphics
Discovered theorems on point H
Concyclic points: EFGH
Sets of parallel and perpendicular lines:
• ABE ⊥ AD
• ACGH ∥ EF ⊥ BD
• AF ⊥ BH ∥ DEG
• BCF ⊥ CD
• BG ∥ DFH ⊥ CE
Congruent segments:
• AG = CH = GH
• AH = CG
• BG = BH = DG = DH
• EG = FH
• EH = FG
OK
Input:


GeoGebra Classic 5
Algebra
poly1 = 13.87
E = (0.92, 1.31)
F = (1.05, 3.94)
j: −1.12x + 4.01y = 14.62
k: 1.51x + 3.88y = 6.47
l: −5.26x + 0.26y = 2.44
G = (−0.37, 1.81)
H = (−0.29, 3.57)
c: x² + y² − 1.39x − 5.28y =
m = 1.39
n = 1.39
p = 3.51
q = 3.51
r = 2.78
s = 2.78
t = 2.78
a = 2.78
b = 2.56
d = 2.56
e = 1.76
f1 = 1.76
g1 = 1.76
h1: −3.88x + 1.51y = 1.88
i1: −1.01x − 2.59y = −8.94
j1: 4.01x + 1.12y = 5.16
k1: 0.75x − 2.67y = −5.13
l1: −2.63x + 0.13y = −2.25
m1: −0.26x − 5.26y = −14.0
CAS
2 Let poly1 be the regular 4-gon with vertices A, B, C
3 Let f be the segment A, B.
4 Let g be the segment B, C.
5 Let E be the midpoint of f.
6 Let F be the midpoint of g.
7 Let j be the line D, F.
8 Let k be the line D, E.
9 Let l be the line A, C.
10 Let G be the intersection of l and k.
11 Let H be the intersection of j and l.
12 Prove that g_1 ≟ e.
13 The statement is true under some non-degeneracy
14 We prove this by contradiction.
15 Let free point A be denoted by (v1,v2).
16 Let free point B be denoted by (v3,v4).
Graphics
l
C
f1
F
H
k
D
B
j
e
G
E
g1
A



GeoGebra Classic 5
Algebra
poly1 = 13.87
E = (0.92, 1.31)
F = (1.05, 3.94)
j: −1.12x + 4.01y = 14.62
k: 1.51x + 3.88y = 6.47
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G = (−0.37, 1.81)
H = (−0.29, 3.57)
c: x² + y² − 1.39x − 5.28y =
m = 1.39
n = 1.39
p = 3.51
q = 3.51
r = 2.78
s = 2.78
t = 2.78
a = 2.78
b = 2.56
d = 2.56
e = 1.76
f1 = 1.76
g1 = 1.76
h1: −3.88x + 1.51y = 1.88
i1: −1.01x − 2.59y = −8.94
j1: 4.01x + 1.12y = 5.16
k1: 0.75x − 2.67y = −5.13
l1: −2.63x + 0.13y = −2.25
m1: −0.26x − 5.26y = −14.0
CAS
55 s9:-v13*v6+v14*v5=0
→ s9 : −v13 v6 + v14 v5 = 0
56 s10:-v13*v10+v14*v9+v13*v8-v9*v8-v14*v7+v10*
→ s10 : −v10 v13 + v10 v7 + v13 v8 − v14 v7
57 s11:-v15*v12+v16*v11+v15*v8-v11*v8-v16*v7+v1
→ s11 : v11 v16 − v11 v8 − v12 v15 + v12 v7
58 s12:-v15*v6+v16*v5=0
→ s12 : −v15 v6 + v16 v5 = 0
59 s13:-1-v51*v16^2-v51*v15^2+2*v51*v16*v14+2*
→ s13 : −v15² v51 − v16² v51 + 2 v13 v15 v5
60 Now we consider the following expression:
61 s1*(-v15^2*v51+1/3*v15*v16*v51+2/3*v15*v51*v
→ 1 = 0
62 Contradiction! This proves the original statement
63 The statement has a difficulty of degree 3.
Graphics
A
B
C
D
E
F
G
H
e
f1
g1
j
k
l

sp-kuwait.ggb

Algebra

- $A = (-0.46, 0.06)$
- $B = (2.3, 2.56)$
- $f = 3.72$
- $poly1 = 13.87$
- $E = (0.92, 1.31)$
- $F = (1.05, 3.94)$
- $j: 1.51x + 3.88y = 6.47$
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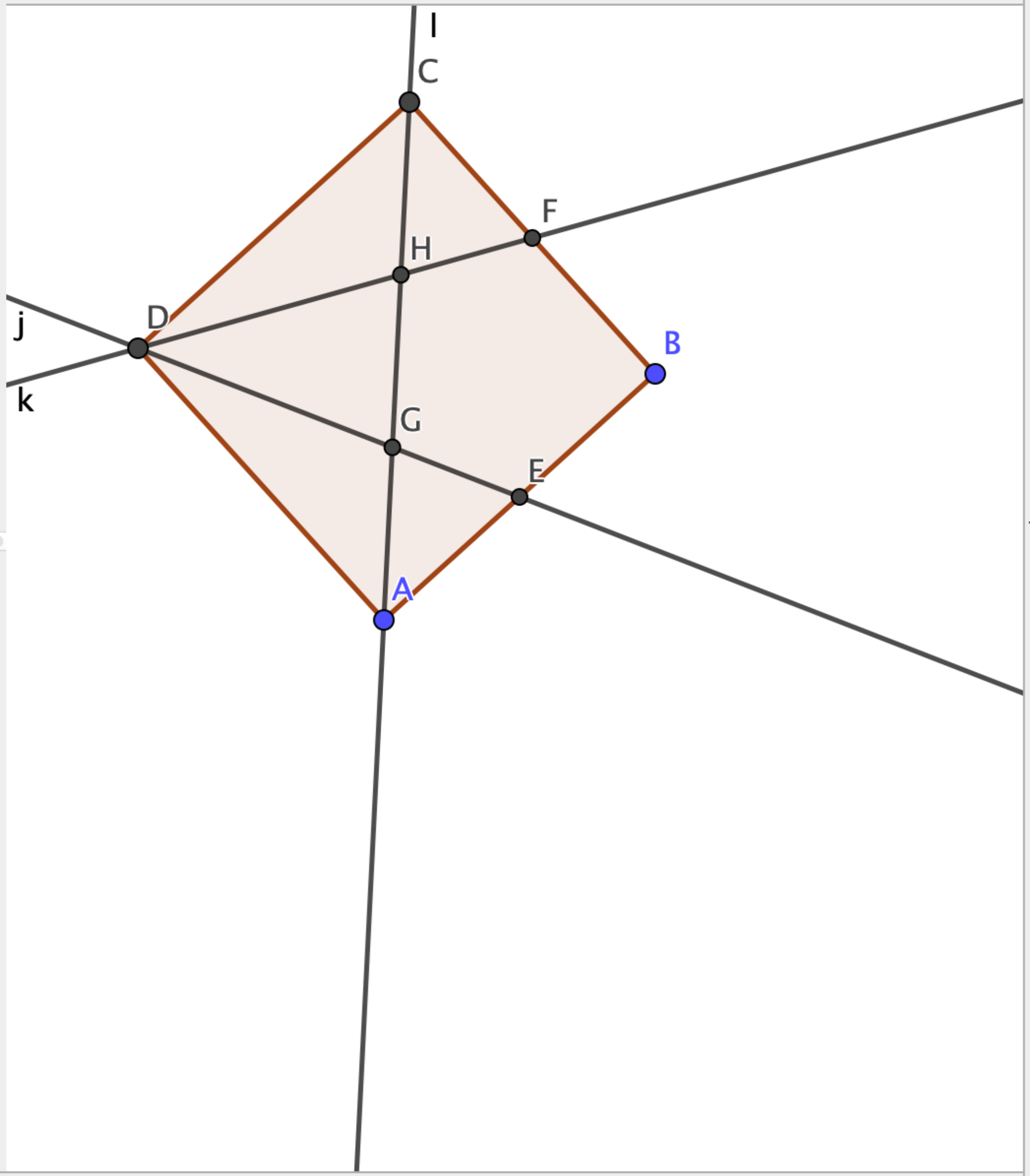
- $l: -5.26x + 0.26y = 2.44$
- $G = (-0.37, 1.81)$
- $H = (-0.29, 3.57)$

CAS

1	Let A, B be arbitrary points.
2	Let poly1 be the regular 4-gon with vertices A, B, C
3	Let f be the segment A, B.
4	Let g be the segment B, C.
5	Let E be the midpoint of f.
6	Let F be the midpoint of g.
7	Let j be the line D, E.
8	Let k be the line D, F.
9	Let l be the line A, C.
10	Let G be the intersection of j and l.
11	Let H be the intersection of l and k.
12	Prove that Distance(G, H) $\stackrel{?}{=}$ Distance(H, C).
13	The statement is true under some non-degeneracy
14	We prove this by contradiction.
15	Let free point A be denoted by (v70,v71).

Graphics



sp-kuwait.ggb

Algebra

A = (−0.46, 0.06)
B = (2.3, 2.56)
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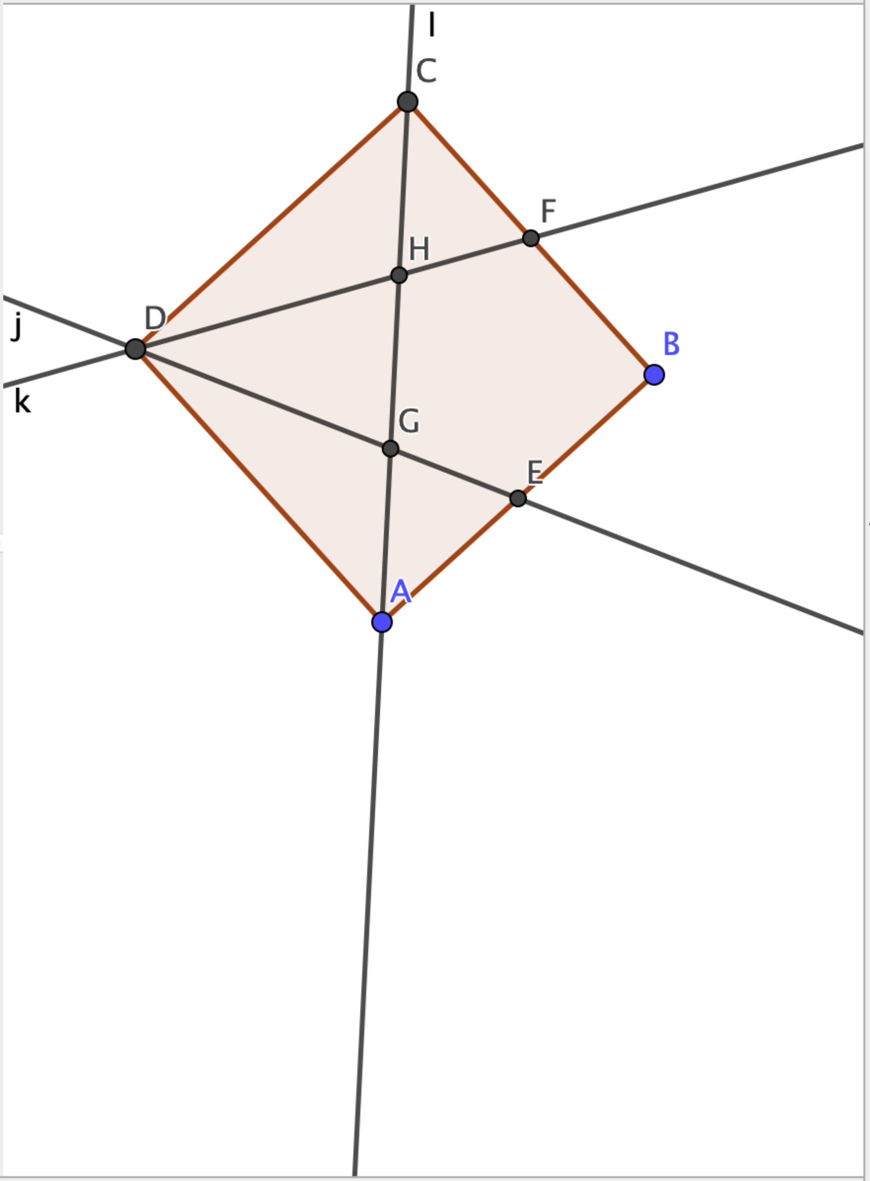
H = (−0.29, 3.57)

CAS

55	s9:-v82*v79+v83*v78+v82*v77-v78*v77-v83*v76+v79*v76= → $s9 : v76\ v79 - v76\ v83 - v77\ v78 + v77\ v82 + v78\ v$
56	s10:-v82*v75+v83*v74=0 → $s10 : v74\ v83 - v75\ v82 = 0$
57	s11:-v84*v75+v85*v74=0 → $s11 : v74\ v85 - v75\ v84 = 0$
58	s12:-v84*v81+v85*v80+v84*v77-v80*v77-v85*v76+v81*v76 → $s12 : v76\ v81 - v76\ v85 - v77\ v80 + v77\ v84 + v80$
59	s13:-1-2*v86*v85*v83+v86*v83^2-2*v86*v84*v82+v86*v82 → $s13 : -v74^2\ v86 - v75^2\ v86 + v82^2\ v86 + v83^2\ v86 +$
60	Now we consider the following expression:
61	s1*(v83*v86*v82+1/3*v86*v82^2-2*v83*v86*v84-2/3*v86*v → $1 = 0$
62	**Contradiction! This proves the original statement.**
63	The statement has a difficulty of degree 3.

Graphics



$A = (-7.97, 1.92)$

$B = (-4.03, -1.38)$

f : PerpendicularBisector(A, B)

$\rightarrow -3.94x + 3.3y = 24.53$

C = Point(f)

$\rightarrow (-4.32, 2.28)$

g = Segment(A, C)

$\rightarrow 3.67$

h = Segment(B, C)

$\rightarrow 3.67$

Input...

1 Let A, B be arbitrary points.

2 Let f be the perpendicular bisector of AB.

3 Let C be a point on f.

4 Let g be the segment A, C.

5 Let h be the segment B, C.

6 Prove that $g \doteq h$.

7 The statement is always true.

8 We prove this by contradiction.

9 Let free point A be denoted by (v1,v2).

10 Let free point B be denoted by (v3,v4).

11 Considering definition f = PerpendicularBisector[A, B]:

12 $e1 : 2\,v5 - v3 - v1 = 0$

$\rightarrow \mathbf{e1 : -v1 - v3 + 2\,v5 = 0}$

13 $e2 : 2\,v6 - v4 - v2 = 0$

$\rightarrow \mathbf{e2 : -v2 - v4 + 2\,v6 = 0}$

14 $e3 : v8 - v6 - v5 + v1 = 0$

$\rightarrow \mathbf{e3 : v1 - v5 - v6 + v8 = 0}$

15 $e4 : v7 + v6 - v5 - v2 = 0$

$\rightarrow \mathbf{e4 : -v2 - v5 + v6 + v7 = 0}$

16 Considering definition C = Point[f]:

17 Let dependent point C be denoted by (v9,v10).

18 $e5 : -v9\,v8 + v10\,v7 + v9\,v6 - v7\,v6 - v10\,v5 + v8\,v5 = 0$

$\rightarrow \mathbf{e5 : -v10\,v5 + v10\,v7 + v5\,v8 - v6\,v7 + v6\,v9 - v8\,v9 = 0}$

19 Thesis reductio ad absurdum (denied statement):

20 $e6 : -1 + 2\,v11\,v10\,v4 - v11\,v4^2 + 2\,v11\,v9\,v3 - v11\,v3^2 - 2\,v11\,v10\,v2 + v11\,v2^2 - 2\,v11\,v9\,v1 + v11\,v1^2 = 0$

$\rightarrow \mathbf{e6 : v11\,v2^2 - v11\,v3^2 - v11\,v4^2 + v1^2\,v11 - 2\,v1\,v11\,v9 - 2\,v10\,v11\,v2 + 2\,v10\,v11\,v4 + 2\,v11\,v3\,v9 - 1 = 0}$

21 Without loss of generality, some coordinates can be fixed:

22 $\{v1 = 0, v2 = 0\}$

$\rightarrow \mathbf{\{v1 = 0, v2 = 0\}}$

23 All hypotheses and the negated thesis after substitutions:

24 $s1 : 2\,v5 - v3 = 0$

$\rightarrow \mathbf{s1 : -v3 + 2\,v5 = 0}$

25 $s2 : 2\,v6 - v4 = 0$

$\rightarrow \mathbf{s2 : -v4 + 2\,v6 = 0}$

26 $s3 : v8 - v6 - v5 = 0$

$\rightarrow \mathbf{s3 : -v5 - v6 + v8 = 0}$

27 $s4 : v7 + v6 - v5 = 0$

$\rightarrow \mathbf{s4 : -v5 + v6 + v7 = 0}$

28 $s5 : -v9\,v8 + v10\,v7 + v9\,v6 - v7\,v6 - v10\,v5 + v8\,v5 = 0$

$\rightarrow \mathbf{s5 : -v10\,v5 + v10\,v7 + v5\,v8 - v6\,v7 + v6\,v9 - v8\,v9 = 0}$

29 $s6 : -1 + 2\,v11\,v10\,v4 - v11\,v4^2 + 2\,v11\,v9\,v3 - v11\,v3^2 = 0$

$\rightarrow \mathbf{s6 : -v11\,v3^2 - v11\,v4^2 + 2\,v10\,v11\,v4 + 2\,v11\,v3\,v9 - 1 = 0}$

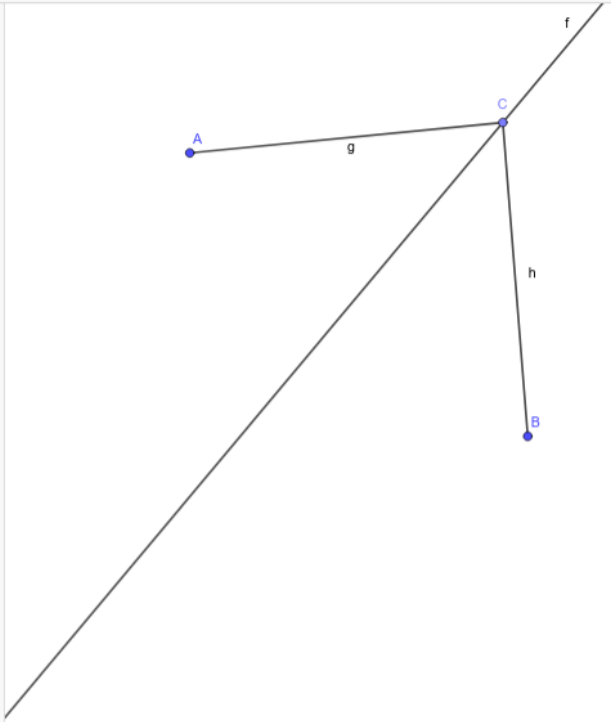
30 Now we consider the following expression:

31 $s1\,(v11\,v3 - v11\,v4 + 2\,v11\,v8 - 2\,v11\,v9) + s2\,(-2\,v10\,v11 + v11\,v3 + v11\,v4 - 2\,v11\,v7) + s3\,(2\,v11\,v3 - 4\,v11\,v9) + s4\,(4\,v10\,v11 - 2\,v11\,v4) + s5\,(-4\,v11) + s6\,(-1)$

$\rightarrow \mathbf{1 = 0}$

32 **Contradiction! This proves the original statement.**

33 The statement has a difficulty of degree 2.



```
> restart:with(PolynomialIdeals):
>
> tesis:=v1^2- 2*v1*v9 - 2*v10*v2 + 2*v10*v4 + v2^2 - v3^2 + 2*v3*v9 - v4^2;-v1-v3+2*v5,-v2-v4+2*v6,v1-v5-v6+v8,-v2-v5+v6+v7, -v10*v5+v10*v7+v5*v8-v6*v7+v6*v9-v8*v9,v11*v2^2-v11*v3^2-v11*v4^2+v11*v1^2-2*v11*v1*v9-2*v10*v11*v2+2*v10*v11*v4+2*v11*v3*v9-1;
```

$$tesis := v1^2 - 2\,v1\,v9 - 2\,v10\,v2 + 2\,v10\,v4 + v2^2 - v3^2 + 2\,v3\,v9 - v4^2$$

$$-v1 - v3 + 2\,v5,\ -v2 - v4 + 2\,v6,\ v1 - v5 - v6 + v8,\ -v2 - v5 + v6 + v7,\ -v10\,v5 + v10\,v7 + v5\,v8 - v6\,v7 + v6\,v9 - v8\,v9,\ v11\,v1^2 - 2\,v11\,v1\,v9 - 2\,v10\,v11\,v2 + 2\,v10\,v11\,v4 + v11\,v2^2 - v11\,v3^2 + 2\,v11\,v3\,v9 - v11\,v4^2 - 1 \quad (1.1)$$

```
> tesis*v11-1;
```

$$\left(v1^2 - 2\,v1\,v9 - 2\,v10\,v2 + 2\,v10\,v4 + v2^2 - v3^2 + 2\,v3\,v9 - v4^2\right) v11 - 1 \quad (1.2)$$

```
> simplify(%-(v11*v2^2-v11*v3^2-v11*v4^2+v11*v1^2-2*v11*v1*v9-2*v10*v11*v2+2*v10*v11*v4+2*v11*v3*v9-1));
```

$$0 \quad (1.3)$$

```
> hipo:=<-v1-v3+2*v5,-v2-v4+2*v6,v1-v5-v6+v8,-v2-v5+v6+v7, -v10*v5+v10*v7+v5*v8-v6*v7+v6*v9-v8*v9>;
```

$$hipo := \langle -v1 - v3 + 2\,v5,\ -v2 - v4 + 2\,v6,\ v1 - v5 - v6 + v8,\ -v2 - v5 + v6 + v7,\ -v10\,v5 + v10\,v7 + v5\,v8 - v6\,v7 + v6\,v9 - v8\,v9\rangle \quad (1.4)$$

```
> tesis in hipo;
```

$$true \quad (1.5)$$

```
> 1 in <-v1 - v3 + 2*v5, -v2 - v4 + 2*v6, v1 - v5 - v6 + v8, -v2 - v5 + v6 + v7, -v10*v5 + v10*v7 + v5*v8 - v6*v7 + v6*v9 - v8*v9, v10*v5+v10*v7+v5*v8-v6*v7+v6*v9-v8*v9,v11*v2^2-v11*v3^2-v11*v4^2+v11*v1^2-2*v11*v1*v9-2*v10*v11*v2+2*v10*v11*v4+2*v11*v3*v9-1>;
```

$$true \quad (1.6)$$

```
> HilbertDimension(hipo);
```

$$5 \quad (1.7)$$

```
> EliminationIdeal(hipo,{v1,v2,v3,v4,v10});EliminationIdeal(hipo,{v1,v2,v3,v4,v9});
```

$$\langle 0\rangle$$

$$\langle 0\rangle \quad (1.8)$$

```
> with(Groebner):  F:=[-v1-v3+2*v5,-v2-v4+2*v6,v1-v5-v6+v8,-v2-v5+v6+v7, -v10*v5+v10*v7+v5*v8-v6*v7+v6*v9-v8*v9]; nops(F);G, C := Basis(F, tdeg(v1,v2,v3,v4,v5,v6,v7,v8,v9,v10), output = extended);nops(C);
```

$$F := [-v1 - v3 + 2\,v5,\ -v2 - v4 + 2\,v6,\ v1 - v5 - v6 + v8,\ -v2 - v5 + v6 + v7,\ -v10\,v5 + v10\,v7 + v5\,v8 - v6\,v7 + v6\,v9 - v8\,v9]$$

$$5$$

$$G, C := [-v5 - v6 + v7 + v4,\ -v5 + v6 - v8 + v3,\ v2 + v5 - v6 - v7,\ v1 - v5 - v6 + v8,\ v10\,v5 - v10\,v7 - v5\,v8 + v6\,v7 - v6\,v9 + v8\,v9], [[0, -1, 0, 1, 0], [-1, 0, -1, 0, 0], [0, 0, 0, -1, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, -1]]$$

$$5 \tag{1.9}$$

```
> NormalForm(tesis, G, tdeg(v1,v2,v3,v4,v5,v6,v7,v8,v9,v10), 'Q');
```

$$0 \tag{1.10}$$

```
> Q;nops(Q);
```

$$[-v4 - v5 - v6 + v7 + 2\,v10,\ -v3 - v5 + v6 - v8 + 2\,v9,\ v2 - v5 + v6 + v7 - 2\,v10,\ v1 + v5 + v6 - v8 - 2\,v9,\ 4]$$

$$5 \tag{1.11}$$

```
> C[1];
```

$$[0, -1, 0, 1, 0] \tag{1.12}$$

```
> C[2];
```

$$[-1, 0, -1, 0, 0] \tag{1.13}$$

```
> C[3];
```

$$[0, 0, 0, -1, 0] \tag{1.14}$$

```
> C[4];
```

$$[0, 0, 1, 0, 0] \tag{1.15}$$

```
> C[5];
```

$$[0, 0, 0, 0, -1] \tag{1.16}$$

```
>
```

```
> add(C[1][i]*F[i],i=1..nops(F))*Q[1]+add(C[2][i]*F[i],
  i=1..nops(F))*Q[2]+add(C[3][i]*F[i],i=1..nops(F))*Q[3]
  +add(C[4][i]*F[i],i=1..nops(F))*Q[4]+add(C[5][i]*F[i],
  i=1..nops(F))*Q[5];simplify(tesis-%);
```

$$(-v5 - v6 + v7 + v4)\,(-v4 - v5 - v6 + v7 + 2\,v10) + (-v5 + v6 - v8 + v3)\,(-v3 - v5 + v6 - v8 + 2\,v9) + (v2 + v5 - v6 - v7)\,(v2 - v5 + v6 + v7 - 2\,v10) + (v1 - v5 - v6 + v8)\,(v1 + v5 + v6 - v8 - 2\,v9) + 4\,v10\,v5 - 4\,v10\,v7 - 4\,v5\,v8 + 4\,v6\,v7 - 4\,v6\,v9 + 4\,v8\,v9$$

$$0 \tag{1.17}$$

Contradiction, complexity 2.

```
> with(Groebner):  FF:=[-v1 - v3 + 2*v5, -v2 - v4 + 2*v6, v1 - v5 - v6 + v8, -v2 - v5 + v6 +
  v7, -v10*v5 + v10*v7 + v5*v8 - v6*v7 + v6*v9 - v8*v9, v10*v5+v10*v7+v5*v8-v6*v7+v6*v9-v8*v9,
  v11*v2^2-v11*v3^2-v11*v4^2+v11*v1^2-2*v11*v1*v9-2*v10*v11*v2+2*v10*v11*v4+2*v11*v3*v9-1];
  nops(FF);GG, CC := Basis(FF, tdeg(v1,v2,v3,v4,v5,v6,v7,v8,v9,v10,v11), output = extended);
  nops(CC);
```

$$FF := [-v1 - v3 + 2\,v5, -v2 - v4 + 2\,v6, v1 - v5 - v6 + v8, -v2 - v5 + v6 + v7, -v10\,v5 + v10\,v7 + v5\,v8 - v6\,v7 + v6\,v9 - v8\,v9, v10\,v5 + v10\,v7 + v5\,v8 - v6\,v7 + v6\,v9 - v8\,v9, v11\,v1^2 - 2\,v11\,v1\,v9 - 2\,v10\,v11\,v2 + 2\,v10\,v11\,v4 + v11\,v2^2 - v11\,v3^2 + 2\,v11\,v3\,v9 - v11\,v4^2 - 1]$$

$$7$$

$$GG, CC := [1], [[v11\,v3 + v11\,v5 - v6\,v11 + v8\,v11 - 2\,v11\,v9, -2\,v11\,v10 + v11\,v4 + v11\,v5 + v6\,v11 - v7\,v11, v1\,v11 + v11\,v3 + 2\,v11\,v5 - 4\,v11\,v9, 4\,v11\,v10 - v2\,v11 - v11\,v4 - 2\,v6\,v11, -4\,v11, 0, -1]]$$

$$1 \tag{2.1}$$

```
> NormalForm(1, GG, tdeg(v1,v2,v3,v4,v5,v6,v7,v8,v9,v10,v11), 'QQ');
```

$$0 \tag{2.2}$$

```
> QQ;nops(QQ);
```

$$[1]$$

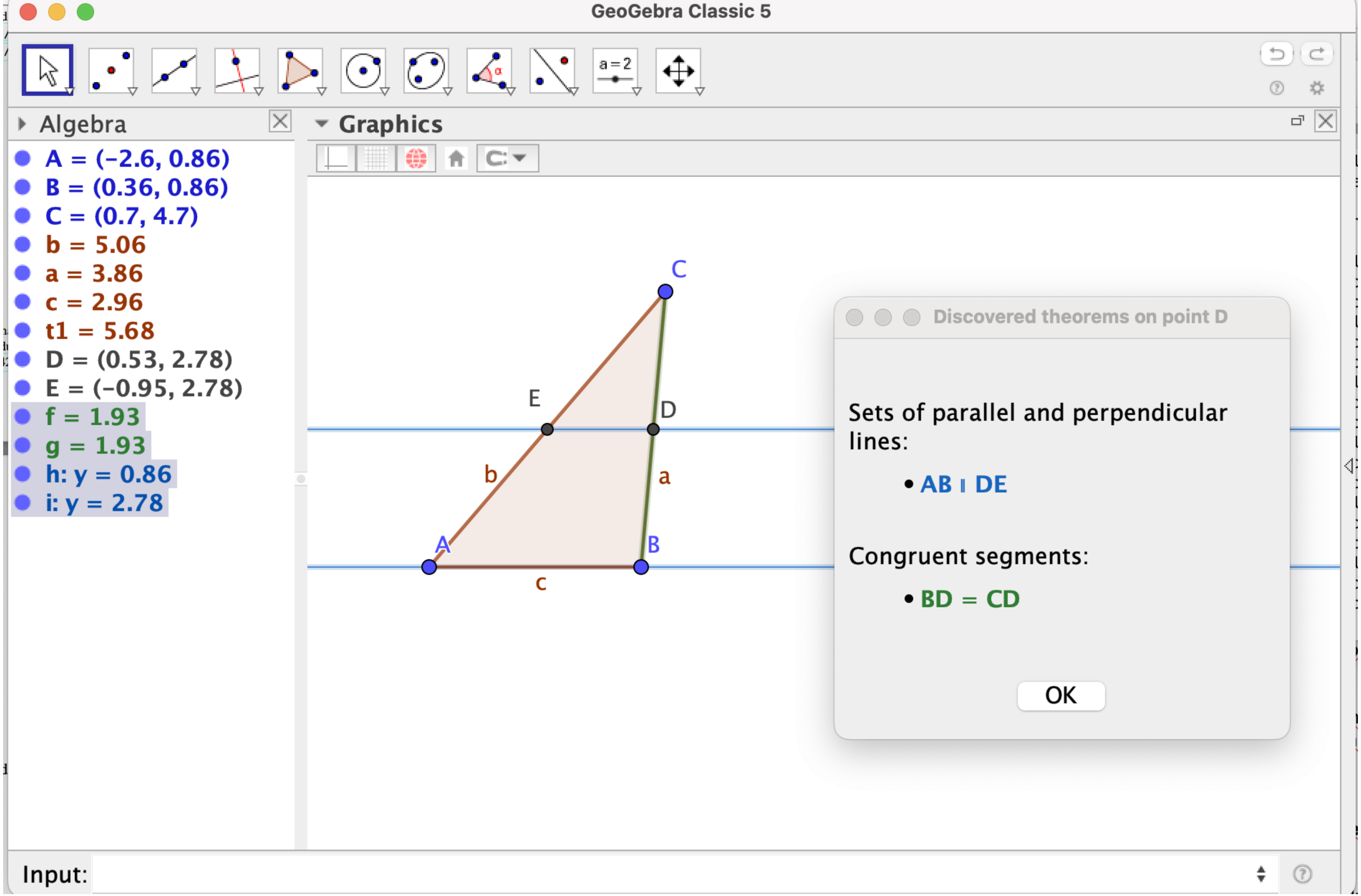
$$1 \tag{2.3}$$

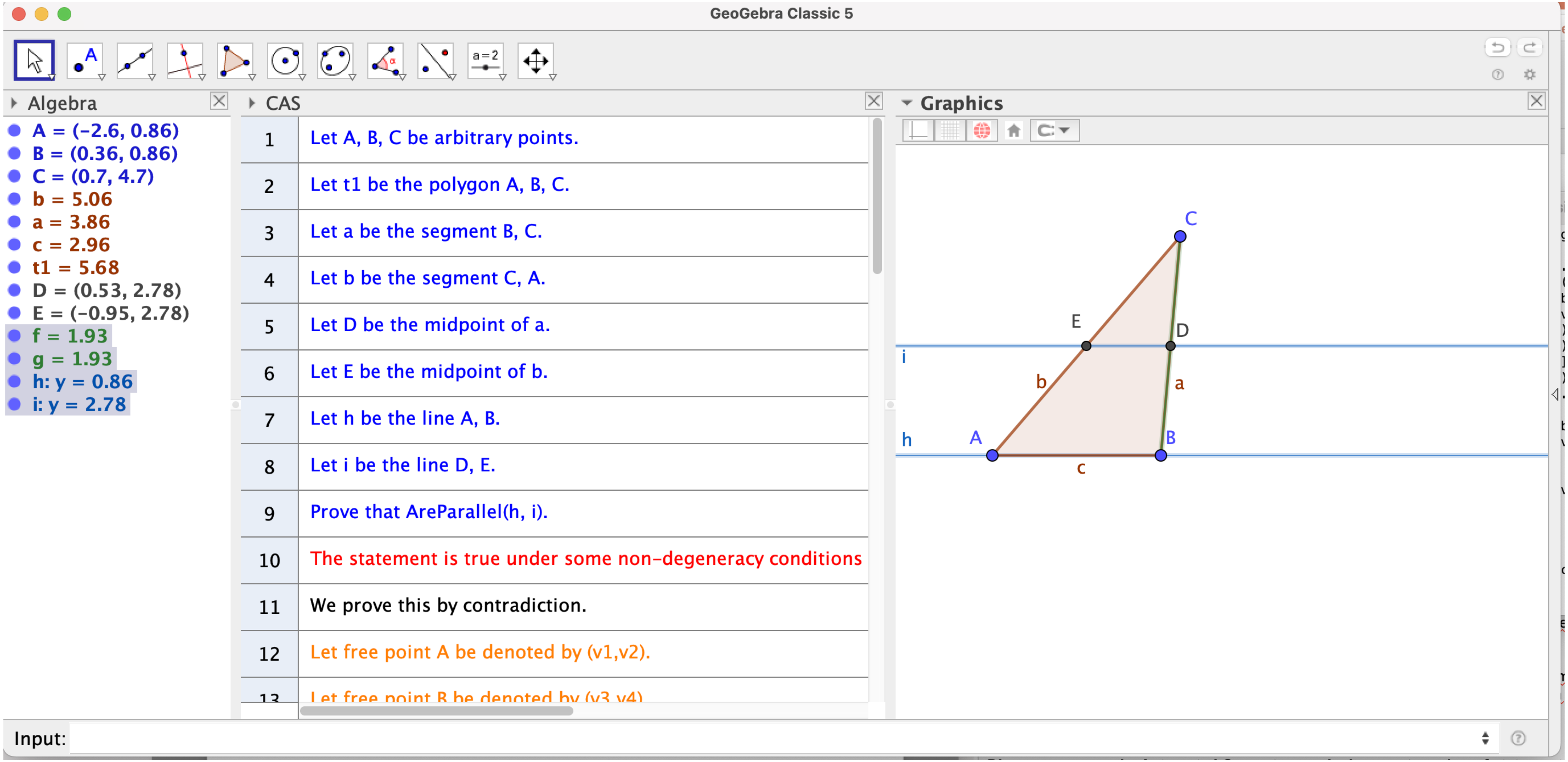
```
> add(CC[1][i]*FF[i],i=1..nops(FF));simplify(1-%);
```

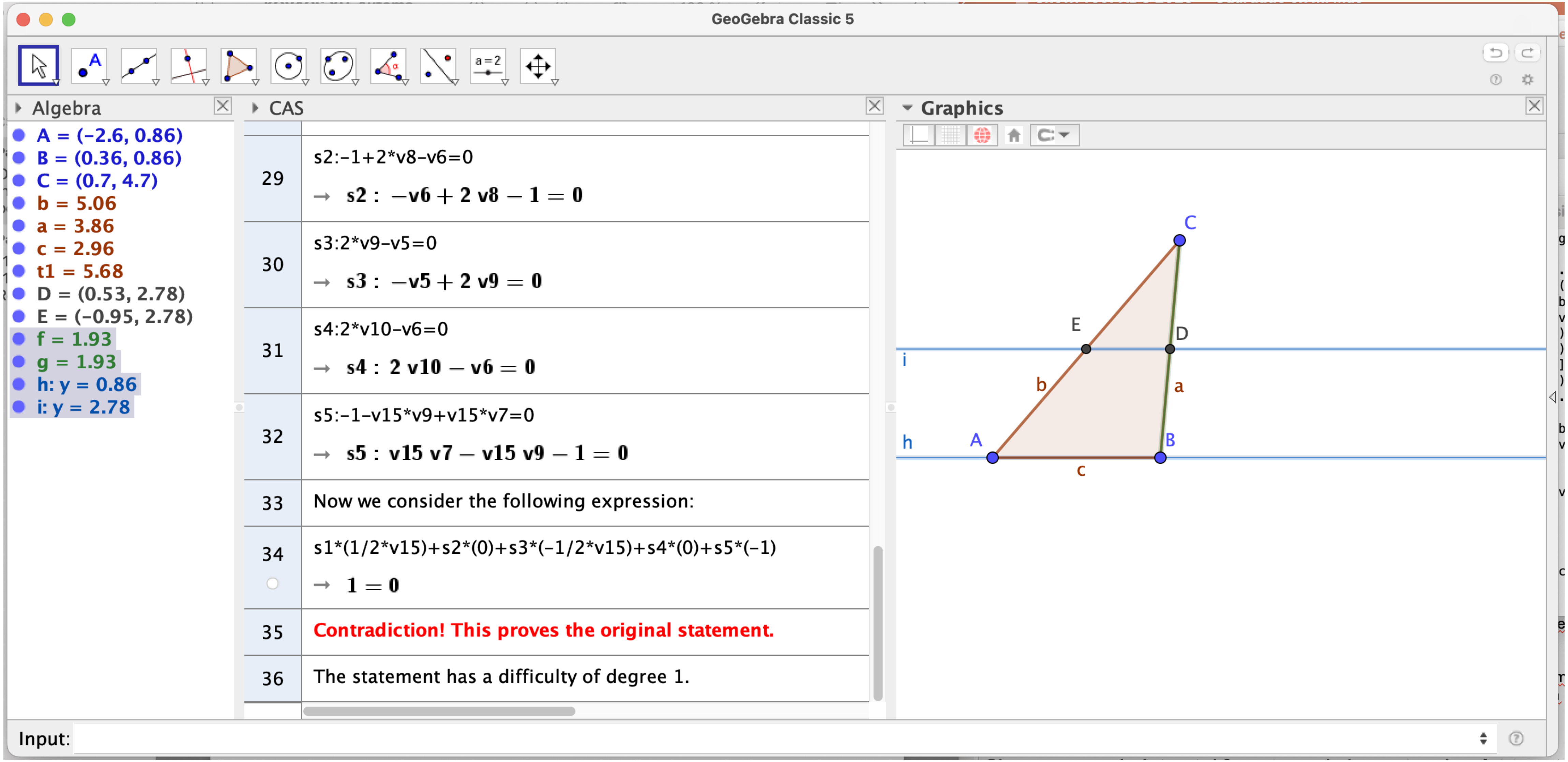
$$(v11\,v3 + v11\,v5 - v6\,v11 + v8\,v11 - 2\,v11\,v9)\,(-v1 - v3 + 2\,v5) + (-2\,v11\,v10 + v11\,v4 + v11\,v5 + v6\,v11 - v7\,v11)\,(-v2 - v4 + 2\,v6) + (v1\,v11 + v11\,v3 + 2\,v11\,v5 - 4\,v11\,v9)\,(v1 - v5 - v6 + v8) + (4\,v11\,v10 - v2\,v11 - v11\,v4 - 2\,v6\,v11)\,(-v2 - v5 + v6 + v7) - 4\,v11\,(-v10\,v5 + v10\,v7 + v5\,v8 - v6\,v7 + v6\,v9 - v8\,v9) - v11\,v1^2 + 2\,v11\,v1\,v9 + 2\,v10\,v11\,v2 - 2\,v10\,v11\,v4 - v11\,v2^2 + v11\,v3^2 - 2\,v11\,v3\,v9 + v11\,v4^2 + 1$$

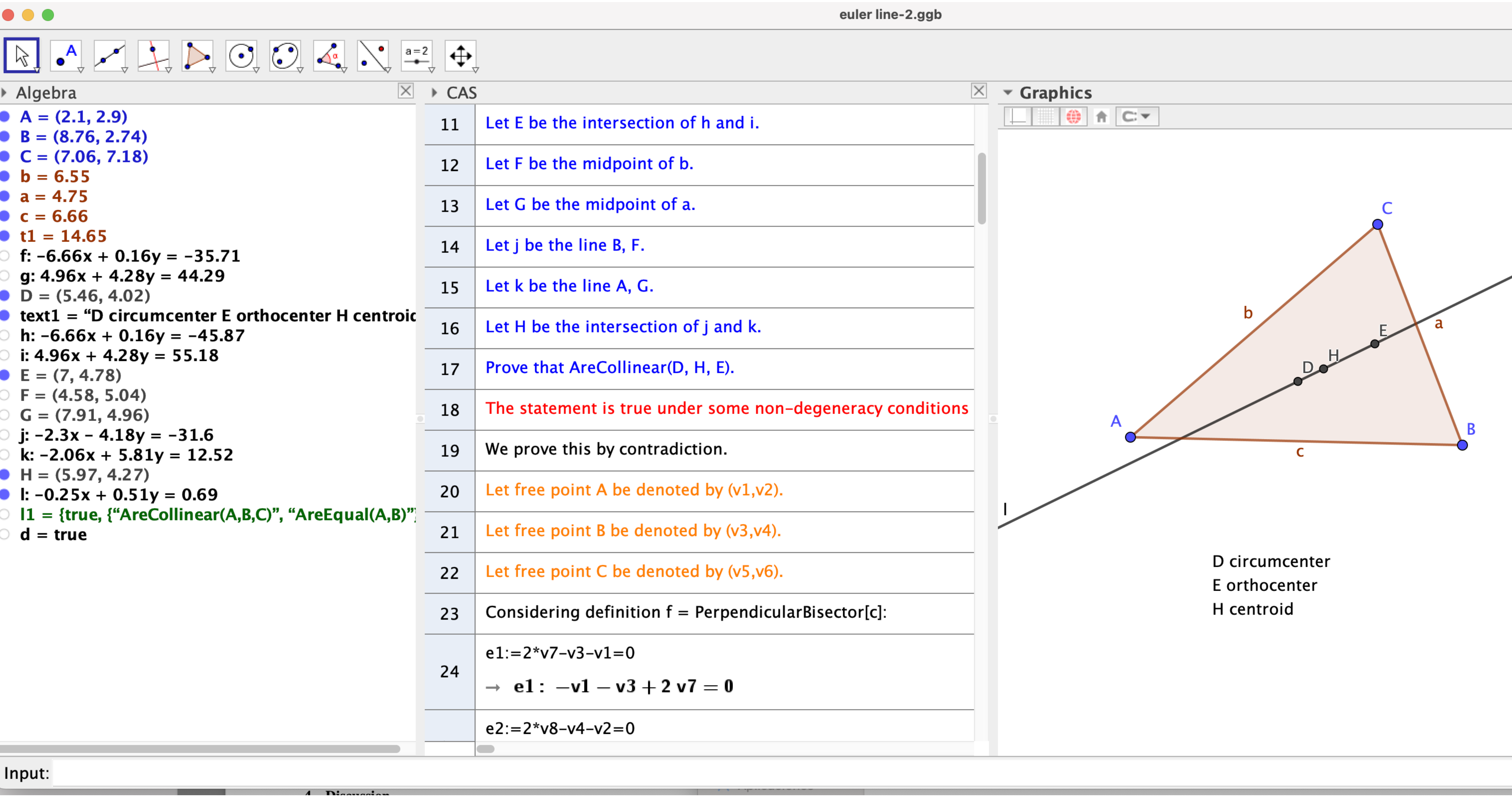
$$0 \tag{2.4}$$

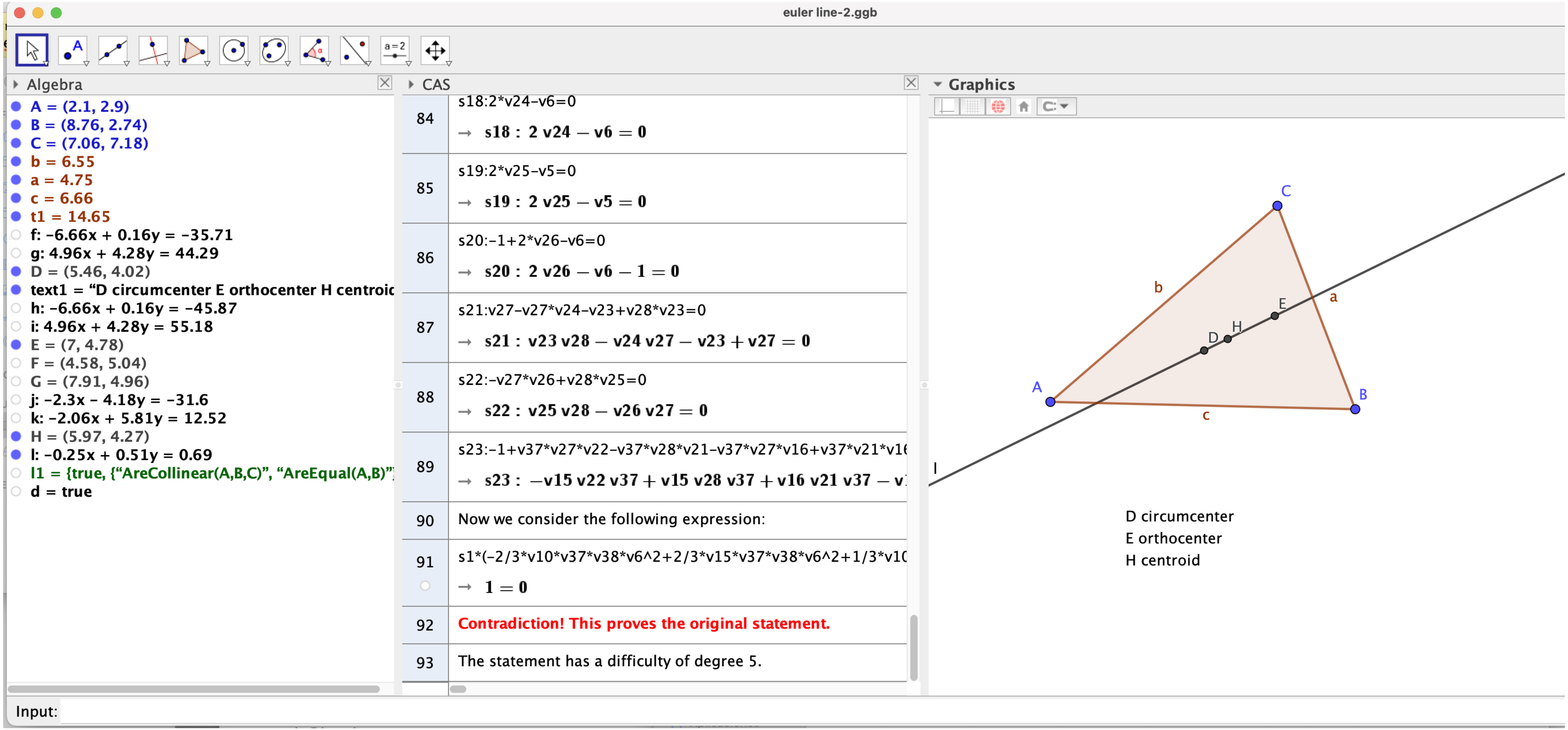
```
>
```


GeoGebra Classic 5
Algebra
Graphics
A = (−2.6, 0.86)
B = (0.36, 0.86)
C = (0.7, 4.7)
b = 5.06
a = 3.86
c = 2.96
t1 = 5.68
D = (0.53, 2.78)
E = (−0.95, 2.78)
f = 1.93
g = 1.93
h: y = 0.86
i: y = 2.78
A
B
C
D
E
a
b
c
Discovered theorems on point D
Sets of parallel and perpendicular lines:
• AB ∥ DE
Congruent segments:
• BD = CD
OK
Input:


GeoGebra Classic 5
Algebra
A = (−2.6, 0.86)
B = (0.36, 0.86)
C = (0.7, 4.7)
b = 5.06
a = 3.86
c = 2.96
t1 = 5.68
D = (0.53, 2.78)
E = (−0.95, 2.78)
f = 1.93
g = 1.93
h: y = 0.86
i: y = 2.78
CAS
1 Let A, B, C be arbitrary points.
2 Let t1 be the polygon A, B, C.
3 Let a be the segment B, C.
4 Let b be the segment C, A.
5 Let D be the midpoint of a.
6 Let E be the midpoint of b.
7 Let h be the line A, B.
8 Let i be the line D, E.
9 Prove that AreParallel(h, i).
10 The statement is true under some non-degeneracy conditions
11 We prove this by contradiction.
12 Let free point A be denoted by (v1,v2).
Graphics
C
E
D
i
b
a
h
A
B
c
Input:


GeoGebra Classic 5
Algebra
A = (−2.6, 0.86)
B = (0.36, 0.86)
C = (0.7, 4.7)
b = 5.06
a = 3.86
c = 2.96
t1 = 5.68
D = (0.53, 2.78)
E = (−0.95, 2.78)
f = 1.93
g = 1.93
h: y = 0.86
i: y = 2.78
CAS
29 s2:−1+2*v8−v6=0
→ s2 : −v6 + 2 v8 − 1 = 0
30 s3:2*v9−v5=0
→ s3 : −v5 + 2 v9 = 0
31 s4:2*v10−v6=0
→ s4 : 2 v10 − v6 = 0
32 s5:−1−v15*v9+v15*v7=0
→ s5 : v15 v7 − v15 v9 − 1 = 0
33 Now we consider the following expression:
34 s1*(1/2*v15)+s2*(0)+s3*(−1/2*v15)+s4*(0)+s5*(−1)
→ 1 = 0
35 Contradiction! This proves the original statement.
36 The statement has a difficulty of degree 1.
Graphics
C
E
D
i
b
a
h
A
B
c
Input:


euler line-2.ggb
a=2
Algebra
A = (2.1, 2.9)
B = (8.76, 2.74)
C = (7.06, 7.18)
b = 6.55
a = 4.75
c = 6.66
t1 = 14.65
f: −6.66x + 0.16y = −35.71
g: 4.96x + 4.28y = 44.29
D = (5.46, 4.02)
text1 = “D circumcenter E orthocenter H centroic
h: −6.66x + 0.16y = −45.87
i: 4.96x + 4.28y = 55.18
E = (7, 4.78)
F = (4.58, 5.04)
G = (7.91, 4.96)
j: −2.3x − 4.18y = −31.6
k: −2.06x + 5.81y = 12.52
H = (5.97, 4.27)
l: −0.25x + 0.51y = 0.69
l1 = {true, {“AreCollinear(A,B,C)”, “AreEqual(A,B)”
d = true
CAS
11 Let E be the intersection of h and i.
12 Let F be the midpoint of b.
13 Let G be the midpoint of a.
14 Let j be the line B, F.
15 Let k be the line A, G.
16 Let H be the intersection of j and k.
17 Prove that AreCollinear(D, H, E).
18 The statement is true under some non−degeneracy conditions
19 We prove this by contradiction.
20 Let free point A be denoted by (v1,v2).
21 Let free point B be denoted by (v3,v4).
22 Let free point C be denoted by (v5,v6).
23 Considering definition f = PerpendicularBisector[c]:
24 e1:=2*v7−v3−v1=0
→ e1 : −v1 − v3 + 2 v7 = 0
e2:=2*v8−v4−v2=0
Graphics
C
b
a
E
H
D
A
B
c
l
D circumcenter
E orthocenter
H centroid
Input:


euler line-2.ggb
Algebra
A = (2.1, 2.9)
B = (8.76, 2.74)
C = (7.06, 7.18)
b = 6.55
a = 4.75
c = 6.66
t1 = 14.65
f: -6.66x + 0.16y = -35.71
g: 4.96x + 4.28y = 44.29
D = (5.46, 4.02)
text1 = “D circumcenter E orthocenter H centroic
h: -6.66x + 0.16y = -45.87
i: 4.96x + 4.28y = 55.18
E = (7, 4.78)
F = (4.58, 5.04)
G = (7.91, 4.96)
j: -2.3x - 4.18y = -31.6
k: -2.06x + 5.81y = 12.52
H = (5.97, 4.27)
l: -0.25x + 0.51y = 0.69
l1 = {true, {“AreCollinear(A,B,C)”, “AreEqual(A,B)”
d = true
CAS
84 s18:2*v24-v6=0
→ s18 : 2 v24 − v6 = 0
85 s19:2*v25-v5=0
→ s19 : 2 v25 − v5 = 0
86 s20:-1+2*v26-v6=0
→ s20 : 2 v26 − v6 − 1 = 0
87 s21:v27-v27*v24-v23+v28*v23=0
→ s21 : v23 v28 − v24 v27 − v23 + v27 = 0
88 s22:-v27*v26+v28*v25=0
→ s22 : v25 v28 − v26 v27 = 0
89 s23:-1+v37*v27*v22-v37*v28*v21-v37*v27*v16+v37*v21*v1
→ s23 : −v15 v22 v37 + v15 v28 v37 + v16 v21 v37 − v
90 Now we consider the following expression:
91 s1*(-2/3*v10*v37*v38*v6^2+2/3*v15*v37*v38*v6^2+1/3*v1
→ 1 = 0
92 Contradiction! This proves the original statement.
93 The statement has a difficulty of degree 5.
Graphics
C
b
a
E
H
D
A
B
c
l
D circumcenter
E orthocenter
H centroid
Input:

sp-9point.ggb

File Edit View Options Tools Window Help

Sign in

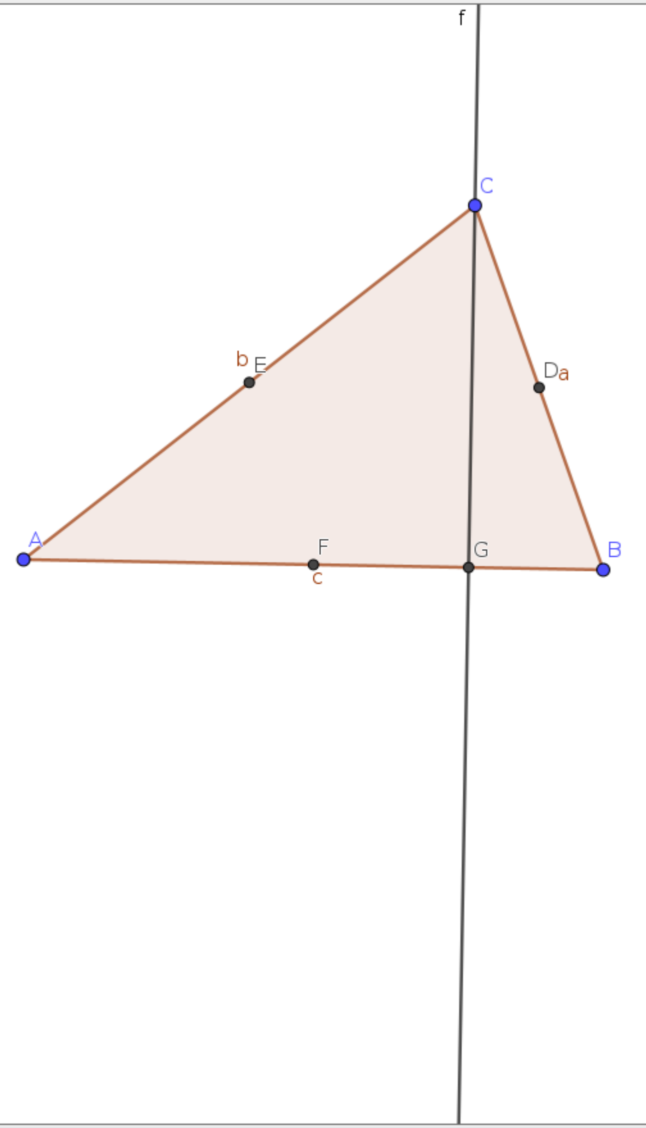
Algebra

- A = (-4.62, -0.56)
- B = (-0.1, -0.64)
- C = (-1.1, 2.2)
- b = 4.47
- a = 3.01
- c = 4.52
- t1 = 6.38
- D = (-0.6, 0.78)
- E = (-2.86, 0.82)
- F = (-2.36, -0.6)
- f: -4.52x + 0.08y = 5.15
- G = (-1.15, -0.62)

CAS

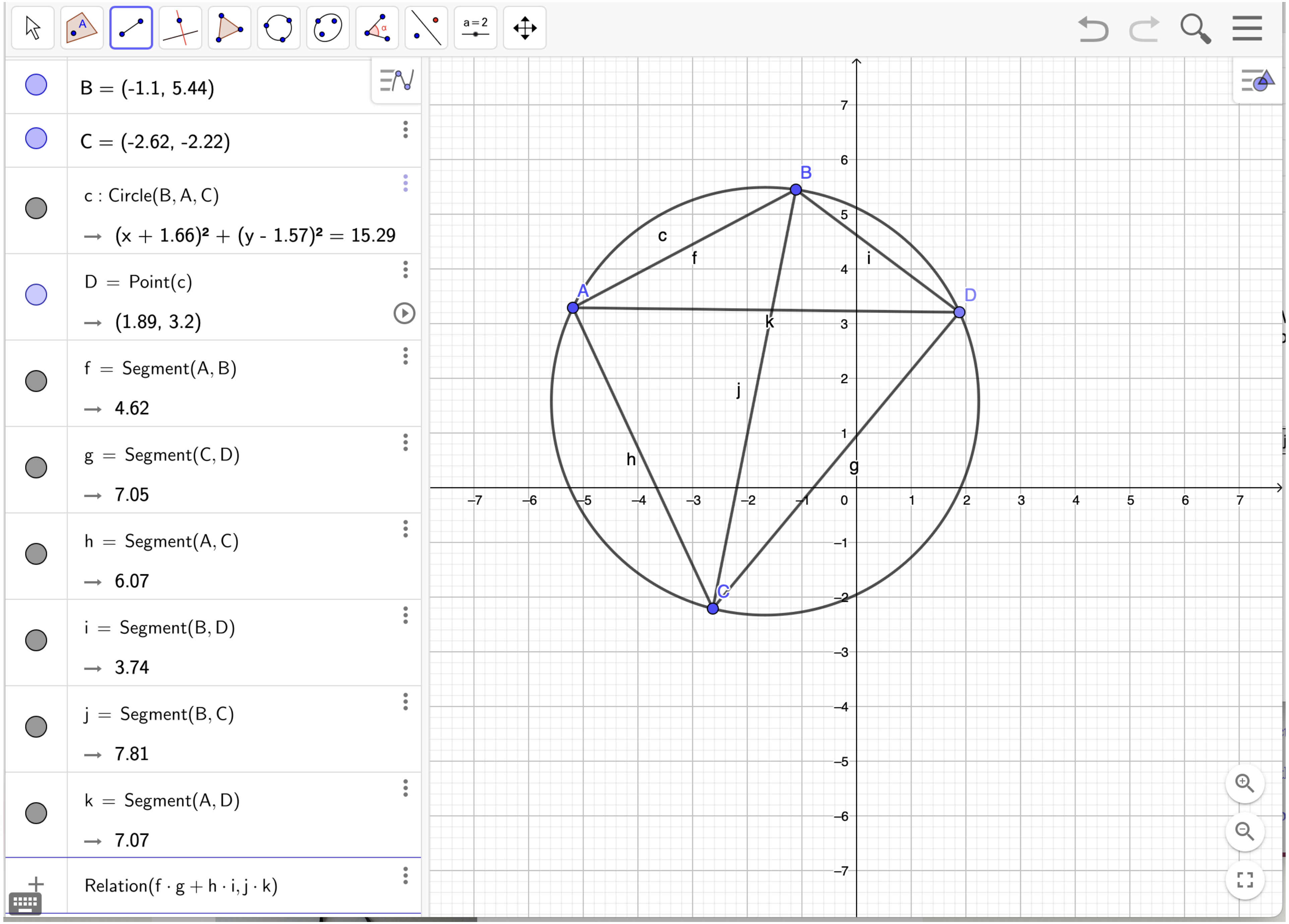
35	s3:-1+2*v8-v6=0 → $s3: -v6 + 2\,v8 - 1 = 0$
36	s4:2*v7-v5=0 → $s4: -v5 + 2\,v7 = 0$
37	s5:2*v10-v6=0 → $s5: 2\,v10 - v6 = 0$
38	s6:-1+2*v12=0 → $s6: 2\,v12 - 1 = 0$
39	s7:-1-v13+v5=0 → $s7: -v13 + v5 - 1 = 0$
40	s8:2*v9-v5=0 → $s8: -v5 + 2\,v9 = 0$
41	s9:2*v11=0 → $s9: 2\,v11 = 0$
42	s10:-v14+v6=0 → $s10: -v14 + v6 = 0$
43	s11:-v15=0 → $s11: -v15 = 0$
44	Now we consider the following expression:
45	s1*(-1)+s2*(1/4*v16*v18*v5-1/8*v18*v5)+s3*(1/2*v9^2*v11*v18-1/2*v8*v10*v11*v18+1 → $1 = 0$
46	Contradiction! This proves the original statement.
47	The statement has a difficulty of degree 4.

Graphics

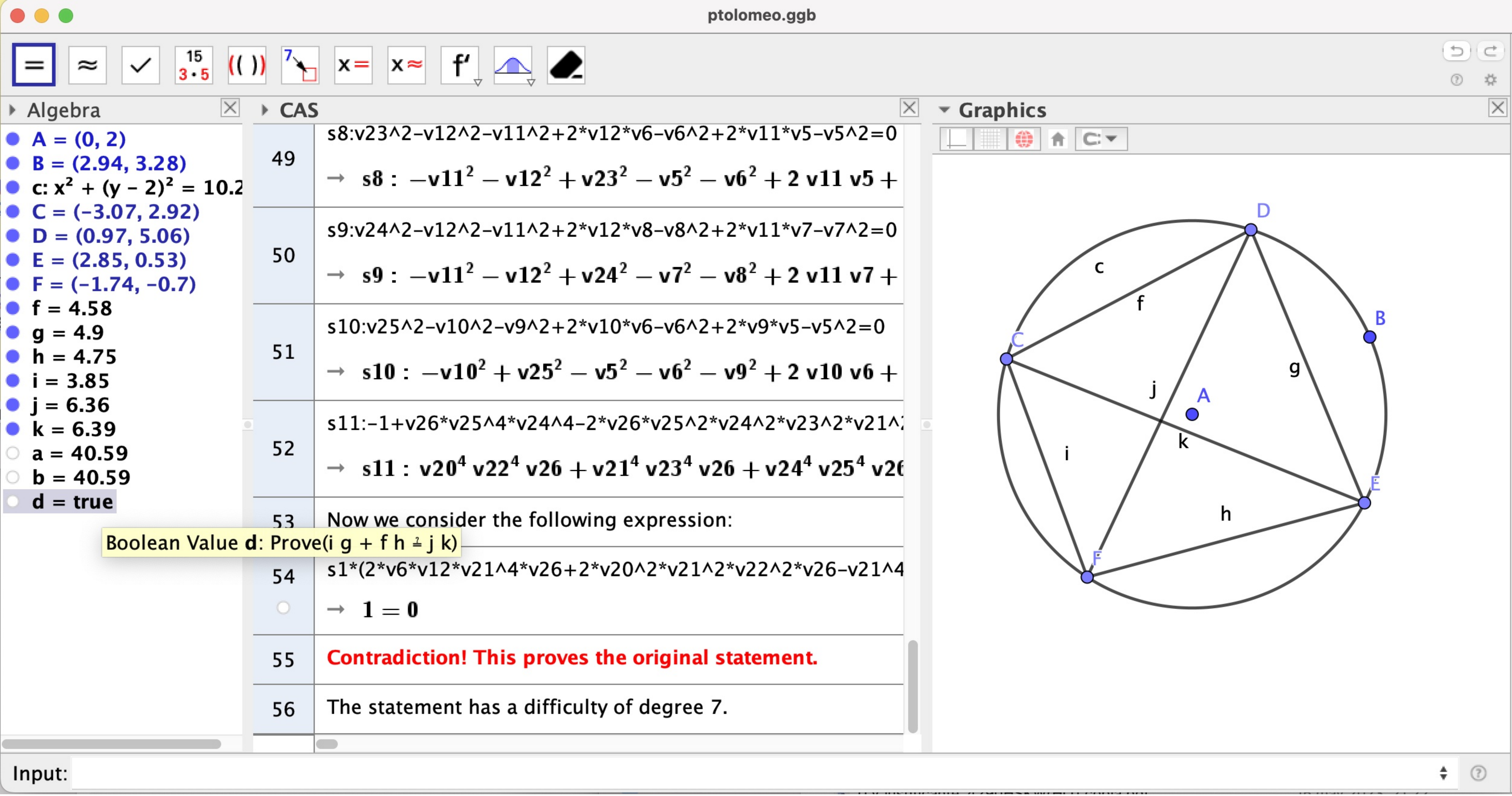


Input:

21:17
2023. 05. 06.



a=2
B = (-1.1, 5.44)
C = (-2.62, -2.22)
c : Circle(B, A, C)
→ (x + 1.66)² + (y - 1.57)² = 15.29
D = Point(c)
→ (1.89, 3.2)
f = Segment(A, B)
→ 4.62
g = Segment(C, D)
→ 7.05
h = Segment(A, C)
→ 6.07
i = Segment(B, D)
→ 3.74
j = Segment(B, C)
→ 7.81
k = Segment(A, D)
→ 7.07
Relation(f · g + h · i, j · k)
A
B
C
D
c
f
g
h
i
j
k


ptolomeo.ggb
Algebra
A = (0, 2)
B = (2.94, 3.28)
c: x² + (y − 2)² = 10.2
C = (−3.07, 2.92)
D = (0.97, 5.06)
E = (2.85, 0.53)
F = (−1.74, −0.7)
f = 4.58
g = 4.9
h = 4.75
i = 3.85
j = 6.36
k = 6.39
a = 40.59
b = 40.59
d = true
Boolean Value d: Prove(i g + f h ≟ j k)
CAS
49 s8:v23^2−v12^2−v11^2+2*v12*v6−v6^2+2*v11*v5−v5^2=0
→ s8 : −v11² − v12² + v23² − v5² − v6² + 2 v11 v5 +
50 s9:v24^2−v12^2−v11^2+2*v12*v8−v8^2+2*v11*v7−v7^2=0
→ s9 : −v11² − v12² + v24² − v7² − v8² + 2 v11 v7 +
51 s10:v25^2−v10^2−v9^2+2*v10*v6−v6^2+2*v9*v5−v5^2=0
→ s10 : −v10² + v25² − v5² − v6² − v9² + 2 v10 v6 +
52 s11:-1+v26*v25^4*v24^4−2*v26*v25^2*v24^2*v23^2*v21^
→ s11 : v20⁴ v22⁴ v26 + v21⁴ v23⁴ v26 + v24⁴ v25⁴ v26
53 Now we consider the following expression:
54 s1*(2*v6*v12*v21^4*v26+2*v20^2*v21^2*v22^2*v26−v21^4
→ 1 = 0
55 Contradiction! This proves the original statement.
56 The statement has a difficulty of degree 7.
Graphics
A
B
C
D
E
F
c
f
g
h
i
j
k
Input:

54. Österreichische Mathematik-Olympiade

Regionalwettbewerb für Fortgeschrittene

30. März 2023

1. Es seien a, b und c reelle Zahlen mit $0 \leq a, b, c \leq 2$. Man beweise, dass

$$(a-b)(b-c)(a-c) \leq 2$$

gilt, und man gebe an, wann Gleichheit eintritt.

(Karl Czakler)

2. Sei $ABCD$ eine Raute mit $\sphericalangle BAD < 90°$. Der Kreis durch D mit Mittelpunkt A schneide die Gerade CD ein zweites Mal im Punkt E. Der Schnittpunkt der Geraden BE und AC sei S.

Man beweise, dass die Punkte A, S, D und E auf einem Kreis liegen.

(Karl Czakler)

3. Man bestimme alle natürlichen Zahlen $n \geq 2$, für die es zwei Anordnungen $(a_1, a_2, \ldots, a_n)$ und $(b_1, b_2, \ldots, b_n)$ der Zahlen $1, 2, \ldots, n$ gibt, sodass $(a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$ aufeinander folgende natürliche Zahlen sind.

(Walther Janous)

4. Man bestimme alle Paare (x, y) von positiven ganzen Zahlen, sodass für $d = \mathrm{ggT}(x, y)$ die Gleichung

$$xyd = x + y + d^2$$

gilt.

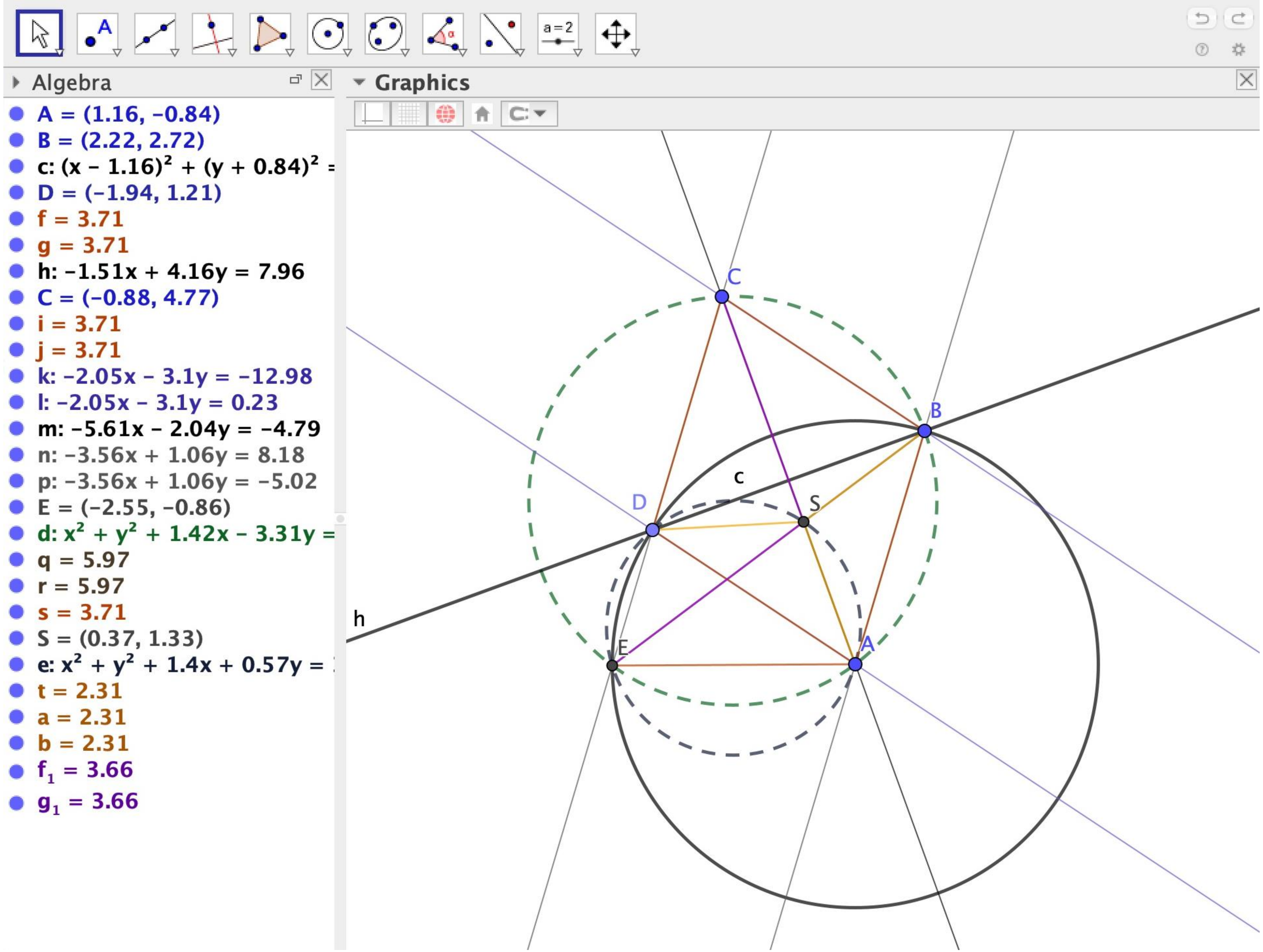
(Walther Janous)

Arbeitszeit: 4 Stunden.
Bei jeder Aufgabe können 8 Punkte erreicht werden.

- Sei ABCD eine Raute mit 4BAD < 90'. Der Kreis durch D mit Mitteipunkt A schneide die Gerade CD ebzweites MaI im Punkt,E. Der Schnittpunkt der Geradet BE und AC sei S.

  Man beweise, dass die Punkte A, S, D und E auf einem Kreis liegen

- Let ABCD be a rhombus with∕BAD < 90◦. The circle through D with center A intersects straight line CD a second time at point E. The intersection of the lines BE and AC is S.
- Prove that the points A, S, D and E lie on a circle.


Algebra
Graphics
A = (1.16, −0.84)
B = (2.22, 2.72)
c: (x − 1.16)² + (y + 0.84)² =
D = (−1.94, 1.21)
f = 3.71
g = 3.71
h: −1.51x + 4.16y = 7.96
C = (−0.88, 4.77)
i = 3.71
j = 3.71
k: −2.05x − 3.1y = −12.98
l: −2.05x − 3.1y = 0.23
m: −5.61x − 2.04y = −4.79
n: −3.56x + 1.06y = 8.18
p: −3.56x + 1.06y = −5.02
E = (−2.55, −0.86)
d: x² + y² + 1.42x − 3.31y =
q = 5.97
r = 5.97
s = 3.71
S = (0.37, 1.33)
e: x² + y² + 1.4x + 0.57y =
t = 2.31
a = 2.31
b = 2.31
f1 = 3.66
g1 = 3.66
C
B
c
D
S
h
E
A

RWF_2023-2B.ggb

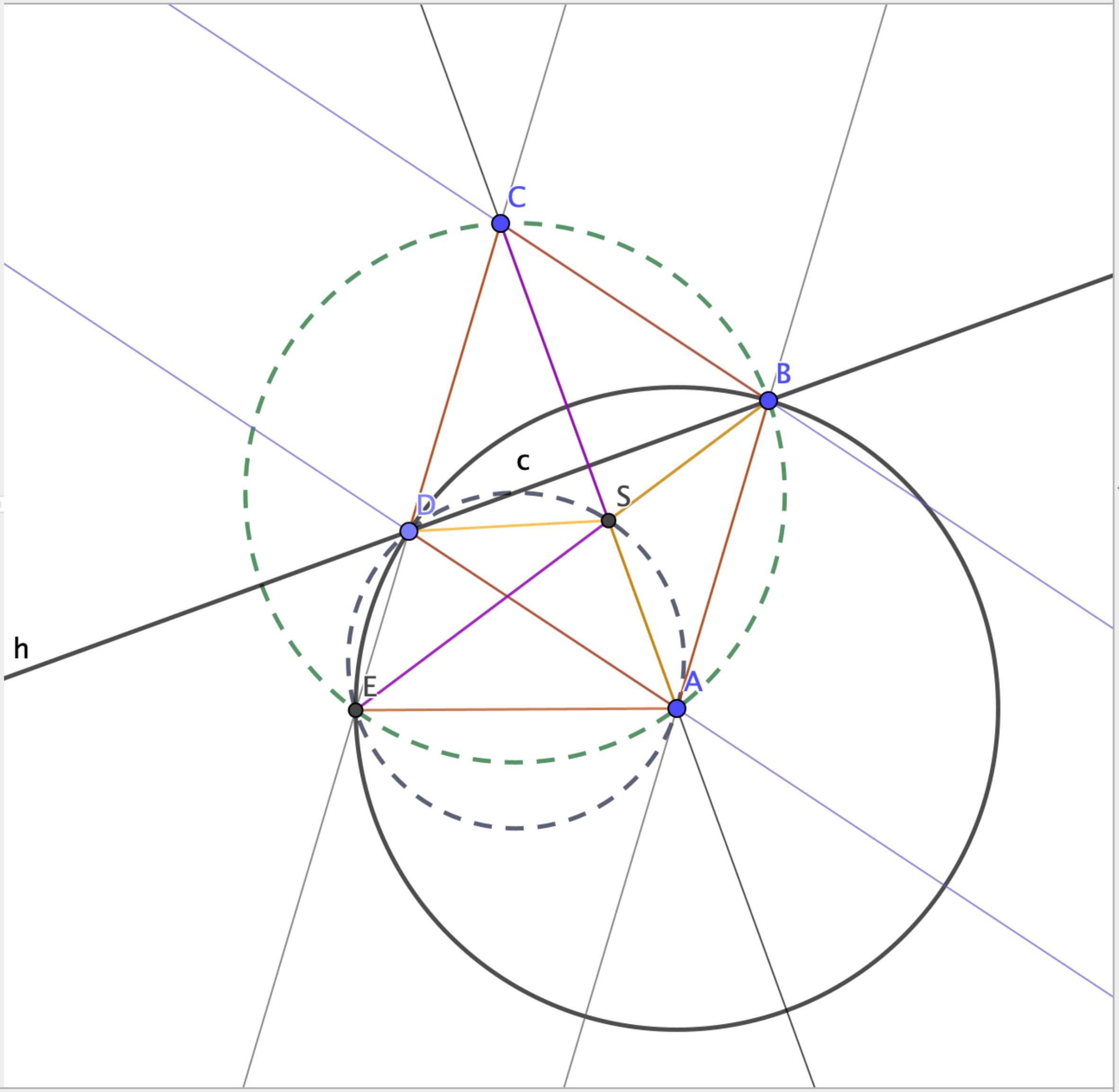
Algebra

- $A = (1.16, -0.84)$
- $B = (2.22, 2.72)$
- $c: (x - 1.16)^2 + (y + 0.84)^2 =$
- $D = (-1.94, 1.21)$
- $f = 3.71$
- $g = 3.71$
- $h: -1.51x + 4.16y = 7.96$
- $C = (-0.88, 4.77)$
- $i = 3.71$
- $j = 3.71$
- $k: -2.05x - 3.1y = -12.98$
- $l: -2.05x - 3.1y = 0.23$
- $m: -5.61x - 2.04y = -4.79$
- $n: -3.56x + 1.06y = 8.18$
- $p: -3.56x + 1.06y = -5.02$
- $E = (-2.55, -0.86)$
- $d: x^2 + y^2 + 1.42x - 3.31y =$
- $q = 5.97$
- $r = 5.97$
- $s = 3.71$
- $S = (0.37, 1.33)$
- $e: x^2 + y^2 + 1.4x + 0.57y =$
- $t = 2.31$
- $a = 2.31$
- $b = 2.31$
- $f_1 = 3.66$
- $g_1 = 3.66$

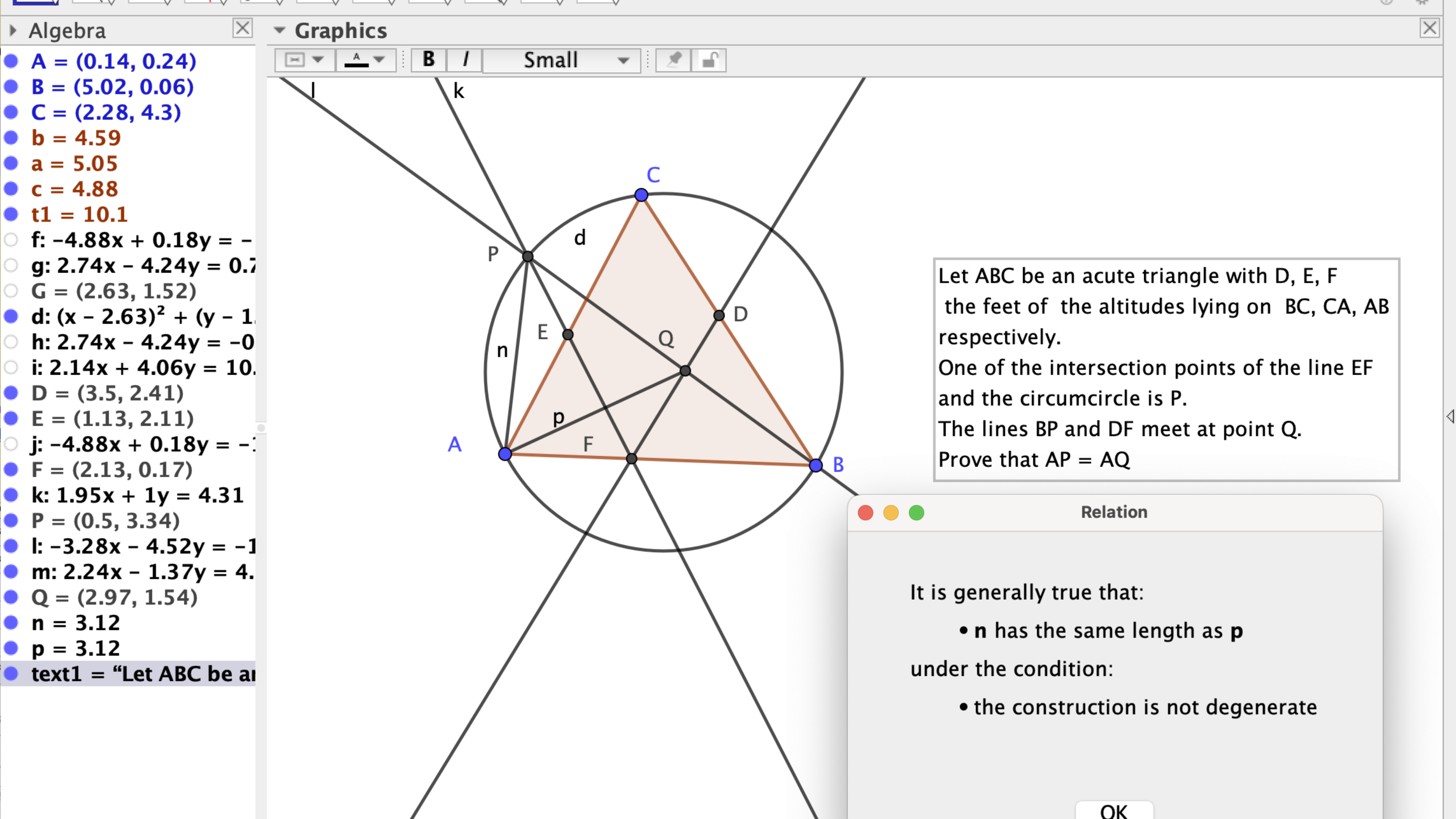
CAS

50	→ $s8: v13\,v6 - v13\,v8 - v14\,v5 + v14\,v7 + v5$
51	s9:1-v14^2-v13^2=0 → $s9: -v13^2 - v14^2 + 1 = 0$
52	s10:-1+v15*v14^2+v15*v13^2-2*v15*v14*v6+v15 → $s10: v15\,v5^2 + v15\,v6^2 + v13^2\,v15 + v14^2\,v$
53	s11:-v16*v8+v17*v7=0 → $s11: -v16\,v8 + v17\,v7 = 0$
54	s12:v16-v16*v14-v13+v17*v13=0 → $s12: v13\,v17 - v14\,v16 - v13 + v16 = 0$
55	s13:-1-v18*v16*v14^2*v6+v18*v17^2*v13*v6+v18 → $s13: -v13\,v17\,v18\,v5^2 - v13\,v17\,v18\,v6^2 +$
56	Now we consider the following expression:
57	s1*(v9*v13^2*v16^2*v18*v19^2+v10*v13*v14*v16 → $1 = 0$
58	Contradiction! This proves the original statement.
59	The statement has a difficulty of degree 9.

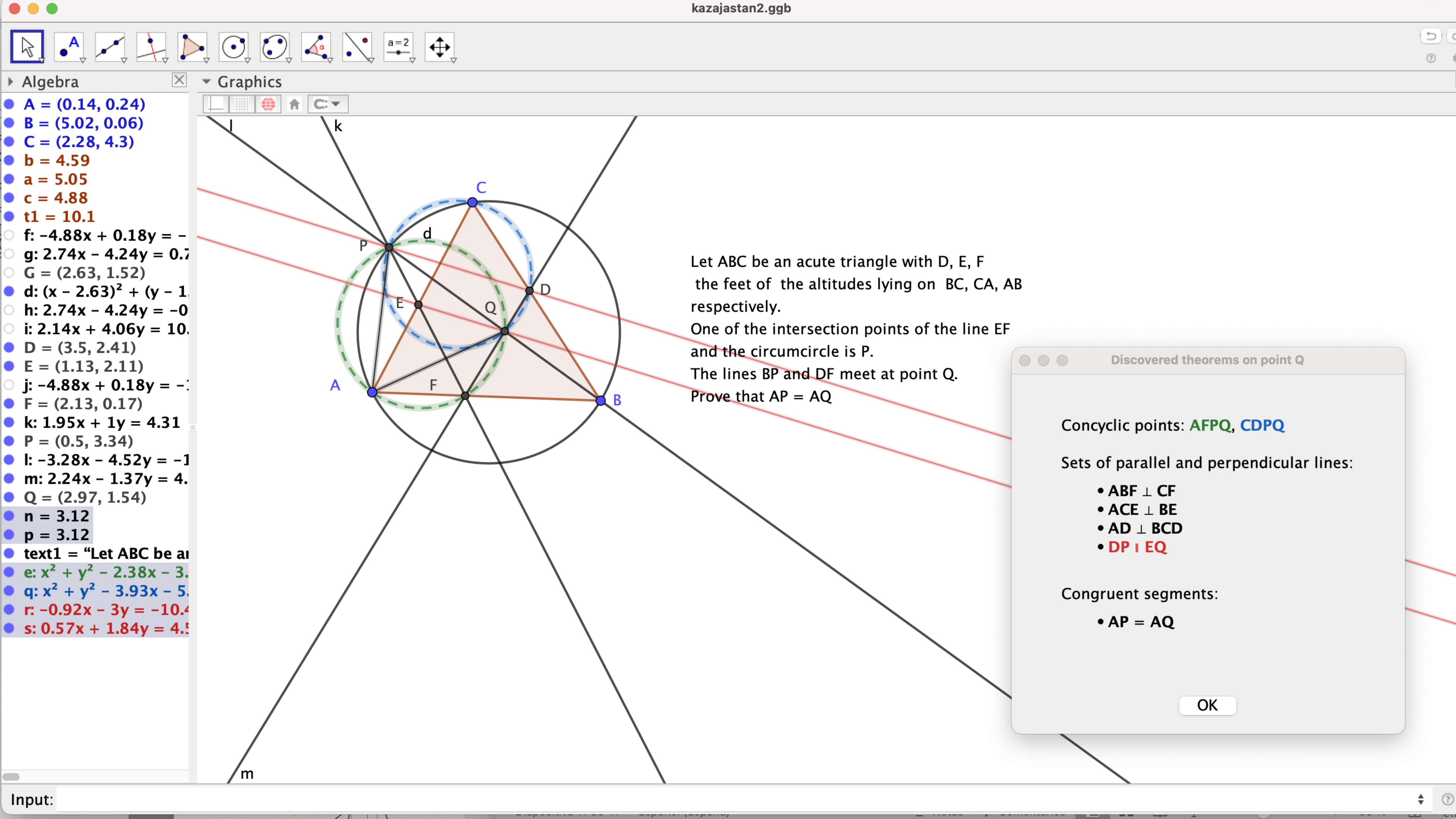
Graphics



Input:



Algebra
A = (0.14, 0.24)
B = (5.02, 0.06)
C = (2.28, 4.3)
b = 4.59
a = 5.05
c = 4.88
t1 = 10.1
f: −4.88x + 0.18y = −
g: 2.74x − 4.24y = 0.7
G = (2.63, 1.52)
d: (x − 2.63)² + (y − 1.
h: 2.74x − 4.24y = −0
i: 2.14x + 4.06y = 10.
D = (3.5, 2.41)
E = (1.13, 2.11)
j: −4.88x + 0.18y = −
F = (2.13, 0.17)
k: 1.95x + 1y = 4.31
P = (0.5, 3.34)
l: −3.28x − 4.52y = −1
m: 2.24x − 1.37y = 4.
Q = (2.97, 1.54)
n = 3.12
p = 3.12
text1 = “Let ABC be a
Graphics
B
I
Small
l
k
C
d
P
D
E
Q
n
p
A
F
B
Let ABC be an acute triangle with D, E, F
the feet of the altitudes lying on BC, CA, AB
respectively.
One of the intersection points of the line EF
and the circumcircle is P.
The lines BP and DF meet at point Q.
Prove that AP = AQ
Relation
It is generally true that:
• n has the same length as p
under the condition:
• the construction is not degenerate
OK


kazajastan2.ggb
Algebra
Graphics
A = (0.14, 0.24)
B = (5.02, 0.06)
C = (2.28, 4.3)
b = 4.59
a = 5.05
c = 4.88
t1 = 10.1
f: −4.88x + 0.18y = −
g: 2.74x − 4.24y = 0.7
G = (2.63, 1.52)
d: (x − 2.63)² + (y − 1.
h: 2.74x − 4.24y = −0
i: 2.14x + 4.06y = 10.
D = (3.5, 2.41)
E = (1.13, 2.11)
j: −4.88x + 0.18y = −
F = (2.13, 0.17)
k: 1.95x + 1y = 4.31
P = (0.5, 3.34)
l: −3.28x − 4.52y = −1
m: 2.24x − 1.37y = 4.
Q = (2.97, 1.54)
n = 3.12
p = 3.12
text1 = "Let ABC be a
e: x² + y² − 2.38x − 3.
q: x² + y² − 3.93x − 5.
r: −0.92x − 3y = −10.4
s: 0.57x + 1.84y = 4.5
Input:
l
k
m
C
d
P
E
Q
D
A
F
B
Let ABC be an acute triangle with D, E, F
the feet of the altitudes lying on BC, CA, AB
respectively.
One of the intersection points of the line EF
and the circumcircle is P.
The lines BP and DF meet at point Q.
Prove that AP = AQ
Discovered theorems on point Q
Concyclic points: AFPQ, CDPQ
Sets of parallel and perpendicular lines:
• ABF ⊥ CF
• ACE ⊥ BE
• AD ⊥ BCD
• DP ∥ EQ
Congruent segments:
• AP = AQ
OK

kazajastan2.ggb

Algebra

- A = (0.14, 0.24)
- B = (5.02, 0.06)
- C = (2.28, 4.3)
- b = 4.59
- a = 5.05
- c = 4.88
- t1 = 10.1
- f: −4.88x + 0.18y = −
- g: 2.74x − 4.24y = 0.7
- G = (2.63, 1.52)
- d: $(x - 2.63)^2 + (y - 1.$
- h: 2.74x − 4.24y = −0
- i: 2.14x + 4.06y = 10.
- D = (3.5, 2.41)
- E = (1.13, 2.11)
- j: −4.88x + 0.18y = −
- F = (2.13, 0.17)
- k: 1.95x + 1y = 4.31
- P = (0.5, 3.34)
- l: −3.28x − 4.52y = −1
- m: 2.24x − 1.37y = 4.
- Q = (2.97, 1.54)
- n = 3.12
- p = 3.12
- text1 = "Let ABC be a

CAS

21 Let n be the segment A, P.

22 Let p be the segment A, Q.

23 Prove that n ≟ p.

24 The statement is true under some non-degeneracy conditions (see below).

25 *Currently no full proof can be provided, but just some steps.*

26 *In the background, all steps are checked, but a full presentation is not yet implemented.*

27 *Please try a newer version of GeoGebra Discovery if possible.*

28 Let free point A be denoted by (v1,v2).

29 Let free point B be denoted by (v3,v4).

30 Let free point C be denoted by (v5,v6).

31 Considering definition f = PerpendicularBisector[c]:

32 e1:=2*v7−v3−v1=0

$\rightarrow\ \mathbf{e1:\ -v1 - v3 + 2\,v7 = 0}$

33 e2:=2*v8−v4−v2=0

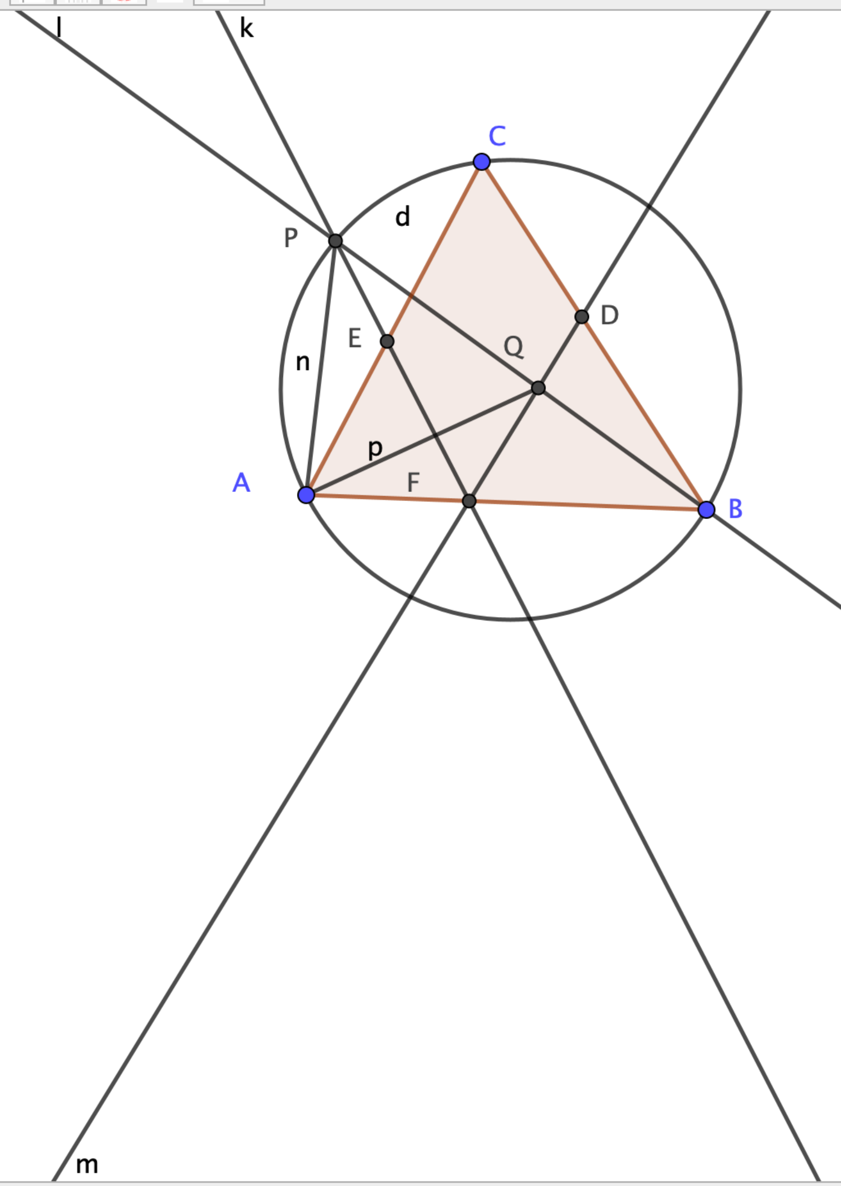
$\rightarrow\ \mathbf{e2:\ -v2 - v4 + 2\,v8 = 0}$

34 e3:=v10−v8−v7+v1=0

$\rightarrow\ \mathbf{e3:\ v1 + v10 - v7 - v8 = 0}$

e4:=v9+v8−v7−v2=0

Graphics



Input:

Prover benchmark for GeoGebra 1

on 2023-08-25 20:23 at kovzol-desktop, Intel(R) Core(TM) i7-4770 CPU @ 3.40GHz

Timeout was set to 20 seconds.

Test file	BotanaGiac F		Botana F	
	Result	Speed	Result	Speed
11gon-1.ggb			9	1000
11gon-2.ggb			10	3752
11gon-3.ggb			9	1372
11gon-4.ggb			9	1821
11gon-5.ggb			8	689
11gon-6.ggb				
11gon-7.ggb			9	10827
11gon-8.ggb				
13gon-1.ggb			10	4342
13gon-2.ggb				
5gon-10gon-1.ggb	-	236	-	198
5gon-10gon-2.ggb	-	141	-	189
7-gon.ggb	-	5289	-	5235
BlazekPech.ggb				
Brianchon-circle-ex19.ggb				
Brianchon-parabola.ggb				
Ceva1.ggb	-	59	-	77
Ceva2.ggb			8	2123
Ceva3.ggb			8	3261
Ceva4.ggb			8	1541
Ceva5.ggb			8	1474
Ceva6.ggb				
Desargues.ggb				
EGMO2012-1.ggb				
Euler-ex162.ggb	-	15913		
EulerLine.ggb	5	115	5	100
FermatPointAngles.ggb				
FermatPointAngles2.ggb				
FermatPointSum.ggb				
Gergonne-ex336.ggb				

circles-are-parallel-ex90.ggb				
circles-chord-half-of-side-ex98.ggb				
circumcenter-of-midpoints-is-midpoint-ex197.ggb	5	152	5	139
circumcenter1.ggb	3	122	2	185
circumcenter2.ggb	-	121	2	185
circumcenter3.ggb	2	111	2	252
circumcenter4.ggb	2	156	2	268
circumcenter5.ggb	2	111	2	154
circumcenter6.ggb	4	91	4	110
circumcircle-3-parabola-tangent.ggb				
construct-perpendicular-line.ggb	-	39	-	47
construct-tangent-to-circle-point.ggb	-	31	-	42
def-line-perpline-perpline.ggb	1	54	1	76
def-points-on-a-circle1.ggb	3	59	3	105
def-points-on-a-circle2.ggb	3	64	3	92
def-points-on-a-line.ggb	1	42	1	73
diameters-are-orthogonal-ex374.ggb	-	65	2	162
dtoc-ex2.ggb	-	68	-	137
dtoc-ex4.ggb	-	172	-	211
ellipse-circle.ggb	1	76	1	119
ellipse-circle2.ggb	-	212	-	397
ellipse-circle3.ggb	-	208	-	290
ellipse-symmetry.ggb	-	357	-	438
ellipse-symmetry2.ggb	-	271	-	382
ex5.3.ggb	2	71	2	90
expression-deltoid1.ggb	-	1271	-	148
expression-deltoid2.ggb	-	142	-	127
expression-ex31.ggb	11	327	5	176
foot-exists.ggb	9	126	5	192
geometric-mean.ggb	-	278	7	204
goldenratio-1.ggb	4	220	4	304
goldenratio-2.ggb				
incenter1.ggb	-	207	-	212
incenter2.ggb			11	390
incenter3.ggb	-	240	-	208
incenter4.ggb	-	460	-	448
intercept.ggb	7	128	7	114

Comments:

Current approach is only good for always true or always geometrically true (not for generally true, that needs to add or multiply times the complexity of the non-degeneracy conditions that have been discovered).

Computation of syzgygies expressing a polynomial in terms of generators, not direct, must be through G Basis. Not trivial, tried different programs.

How to be sure of minimal degrees? No one knows.

Depends greatly of choice of monomial order?

Is there a relation of degree of syzygies and complexity of membership problem (for example, if thesis is in the ideal of hypotheses)?

Is there a relation between the degree of syzygyies in the reductio ad absurdum and the minimal power of the thesis in the radical of hypotheses?

Extend to the case of real equations and inequations.