An Introduction to Interactive Theorem Proving in Geometry using the GeoCoq library

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This tutorial

- A very short introduction to interactive theorem proving using Coq.
- A introduction to the core of GeoCoq library and its tactics.
- Some exercises.
- No prerequisite in logic, nor computer science.
- Intended audience: high-school teachers, researchers in a different field.

Not in this tutorial

- Nothing about numbers, nor induction, nor inductive properties.
- Nothing about how to define data structures (inductive types), nor programs.

To go further: Yves Bertot (May 2010). "Coq in a Hurry".
Yves Bertot and Pierre Castéran (2004).
Interactive Theorem Proving and Program Development, Coq'Art: The Ca
Texts in Theoretical Computer Science. An EATCS Series.

ATP vs ITP

Proof assistants, such as Coq, are not Automated Theorem Prover (ATP), but Interactive Theorem Prover (ITP). The proof assistant checks the proof that the user of the system constructed. Proof assistants do not help finding the proofs, they guaranty that the proof is correct.

GeoCoq inside a browser

https://tinyurl.com/ADG2023

Disclaimer

- Nothing to install !
- But slower than native version.
- Some computationally intensive tactics can produce stack overflow...

Logic	Coq's syntax
false	`False`
true	`True`
$a = b$	`a = b`
$a \neq b$	`a <> b`
not A	`~ A`
A or B	`A \/ B`
A and B	`A /\ B`
A implies B	`A -> B`
A is equivalent to B	`A <-> B`
$f(x, y, z)$	`(f x y z)`
$\forall xy, P(x, y)$	`forall (x:A) (y:B), P x y`
$\exists xy, P(x, y)$	`exists (x:A) (y:B), P x y`

Curryfication

Usually in Coq (and other functional languages or proof assistants), a property:

$$(H_1 \wedge H_2 \wedge \ldots \wedge H_n) \Rightarrow G$$

is written as:

$$H_1 \Rightarrow (H_2 \Rightarrow \ldots \Rightarrow (H_n \Rightarrow G) \ldots)$$

In Coq, implication and function types are both denoted by `->`.

Example

```
Tpoint : Type;
Cong : Tpoint -> Tpoint -> Tpoint -> Tpoint -> Prop;
cong_inner_transitivity : forall A B C D E F,
 Cong A B C D -> Cong A B E F -> Cong C D E F;
```

$$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$$

Definitions and theorems

Example

```
Definition Midpoint M A B :=
       Bet A M B /\ Cong A M M B.

Lemma varignon :
 forall A B C D I J K L,
  A<>C -> B<>D -> ~ Col I J K ->
  Midpoint I A B ->
  Midpoint J B C ->
  Midpoint K C D ->
  Midpoint L A D ->
  Parallelogram I J K L.
```

Basic Coq tactics 1/2

When the goal is . . .	use tactic . . .
`p /\ q`	`split`
`p \/ q`	`left` or `right`
`p -> q`	`intro H`
`~p`	`intro H`
`p <-> q`	`split`
`forall x, p`	`intro x`
`exists x, p`	`exists t`
an assumption	`assumption`
a definition	`unfold`

Basic Coq tactics 2/2

To use hypothesis H . . .	use tactic . . .
`p \/ q`	`destruct H as [H1|H2]`
`p /\ q`	`destruct H as [H1 H2]`
`p -> q`	`apply H`
`p <-> q`	`apply H`
`~p`	`apply H`
`False`	`contradiction`
`forall x, p`	`apply H` or `apply H in`
`exists x, p`	`destruct H as [x G]`

To introduce a new hypothesis H . . .	use tactic . . .
`assert (Hnew: stm).`	
`assert (Hnew:= proof).`	

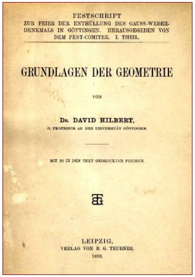
Basic Coq commands

- Check : print the statement (type) of an identifier
- Search foo ”bar” : search in the library lemmas involving foo and with ”bar” in their name.

Outline

GeoCoq

- An Open Source library about foundations of geometry
- Contributors: Michael Beeson, Gabriel Braun, Pierre Boutry, Charly Gries, Julien Narboux, Pascal Schreck
- Size: $>$ 4000 Lemmas, $>$ 130000 lines
- License: LGPL3



FESTSCHRIFT
ZUR FEIER DER ENTHÜLLUNG DES GAUSS-WEBER-DENKMALS IN GÖTTINGEN. HERAUSGEGEBEN VON DEM FEST-COMITEE. I. THEIL.

GRUNDLAGEN DER GEOMETRIE

VON

DR. DAVID HILBERT,
O. PROFESSOR AN DER UNIVERSITÄT GÖTTINGEN.

MIT 50 IN DEN TEXT GEDRUCKTEN FIGUREN.

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VERLAG VON B. G. TEUBNER.
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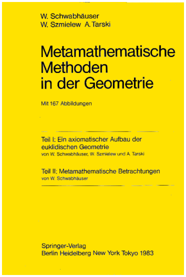


W. Schwabhäuser
W. Szmielew A. Tarski

Metamathematische Methoden in der Geometrie

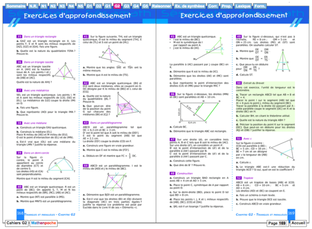
Mit 167 Abbildungen

Teil I: Ein axiomatischer Aufbau der euklidischen Geometrie
von W. Schwabhäuser, W. Szmielew und A. Tarski

Teil II: Metamathematische Betrachtungen
von W. Schwabhäuser

Springer-Verlag
Berlin Heidelberg New York Tokyo 1983

Euclide Hilbert Tarski


Exercices d'approfondissement

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Les cahiers SÉSAMATH

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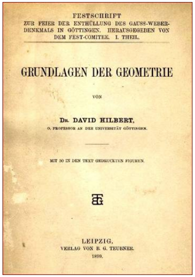
Exercises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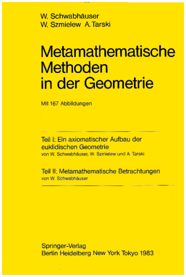
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What we have:

Axiom systems Tarski's, Hilbert's, Euclid's and variants.

Foundations In arbitrary dimension, in neutral geometry. Betweenness, Two-sides, One-side, Collinearity, Midpoint, Symmetric point, Perpendicularity, Parallelism, Angles, Co-planarity, ...

Classic theorems Pappus, Pythagoras, Thales' intercept theorem, Thales' circle theorem, nine point circle, Euler line, orthocenter, circumcenter, incenter, centroid, quadrilaterals, Sum of angles, Varignon's theorem, ...

Arithmetization Coordinates and possibility to use Gröbner basis.

An Euclidean model of Tarski's and Hilbert's axioms using Pythagorean ordered field

High-school Some exercises

What is missing:

- Consequence of continuity: trigonometry, areas
- Model of equal-area axioms (but available in HOL-Light !)
- Model of hyperbolic geometry (but available in Isabelle !)
- Complex geometry (but available in Isabelle !)

Adhoc Tactics 1/2

- `assert_bets` deduces new facts of the form `Bet A B C`. It deduces that `Bet A B C` if :
  - *B* is the midpoint of segment *AC*
- `assert_cols` deduces new facts of the form `Col A B C`. It deduces that `Col A B C` if :
  - Bet *A B C*
  - *A* is the midpoint of segment *BC*
  - *C* is on ray *AB*
  - $AB \parallel CA$
- `assert_ncols` deduces non colinearity from non coplanarity and other predicates
- `assert_diffs` deduces inequality from different assumptions.

Adhoc Tactics 2/2

- `treat_equalities` This tactic looks for any hypotheses of the form $A = B$, it replaces A by B everywhere, it removes hypotheses which become trivial or redundant and it tries to deduce other equalities. For example if $A = B$, and I is midpoint of AB than $A = I$, if $A = B$ and $AB \equiv CD$ then $C = D$, etc
- `perm_apply t` tries to apply the theorem `t` modulo permutations of the predicates (Col, Bet, Par, Perp, Perp_in, is_midpoint,).
- `finish` tries to solve the goal using the tactics to solve permutations and trivial goals based on auto.
- `ColR` uses pseudo-transitivity of colinearity.

An ”axiom free” development

Axiom = global variable, instead we use type classes. Tarski’s geometry is something with a type point and two predicates verifying some axioms.

```
Class Tarski_neutral_dimensionless :=
{
 Tpoint : Type;
 Bet : Tpoint -> Tpoint -> Tpoint -> Prop;
 Cong : Tpoint -> Tpoint -> Tpoint -> Tpoint -> Prop;
 cong_pseudo_reflexivity : forall A B, Cong A B B A;
 cong_inner_transitivity : forall A B C D E F,
   Cong A B C D -> Cong A B E F -> Cong C D E F;
 cong_identity : forall A B C, Cong A B C C -> A = B;
 segment_construction : forall A B C D,
   exists E, Bet A B E /\ Cong B E C D;
   ...
```

Then, we can also formalize some meta-theoretical results:

”Equivalence” between axiom systems:

```
Instance Hilbert_euclidean_follows_from_Tarski_euclidean :
  Hilbert_euclidean
  Hilbert_neutral_follows_from_Tarski_neutral.
```

Defining the ambient theory

If one wants to work in neutral geometry or arbitrary dimension ≥ 2:

```
Section T.
Context `{M:Tarski_neutral_dimensionless}.
Lemma cong_reflexivity : forall A B,
 Cong A B A B.
Proof.
...
Qed.
End T.
```

For Euclidean geometry one can use:

```
Context `{TE:Tarski_euclidean}.
```

Outline

GeoCoq/Euclid in Dedukti

Formalization of Euclid Book 1: 238 lemmas, 20klocs (15% of GeoCoq).
Features: no inductive, no fixpoint, no reflexivity, first-order proofs, simple tactics.
Yoan Geran has exported our formalization of Euclid/Book 1 to: Coq, HOL-Light, Lean, Matita, PVS and Open Theory:

`https://github.com/Karnaj/sttfa_geocoq_euclid`

The (compressed) size of the translated proofs are multiplied by 10 (Lean, Matita, Coq), 25 (Hol-Light) and 50 (PVS).

GeoCoq ported to Isabelle : IsaGeoCoq

Roland Coghetto started to port GeoCoq to Isabelle. First version is available in AFP since 2021 (22klocs):

`https://www.isa-afp.org/entries/IsaGeoCoq.html`

The second version is in preparation:

`https://github.com/CoghettoR/IsaGeoCoq2_R1`

It contains 2850 lemmas, 18 locales and 92klocs making it one of the largest Isabelle contributions (roughly 75% of GeoCoq).

GeoCoq/Tarski ported to Lean

Bhavic Mehta ported 448 lemmas of GeoCoq/Tarski to Lean (roughly 30% of GeoCoq/Tarski, 10% of GeoCoq):

`https://github.com/leanprover-community/mathlib/tree/GeoLean`