Formalization, automatization and visualization of hyperbolic geometry

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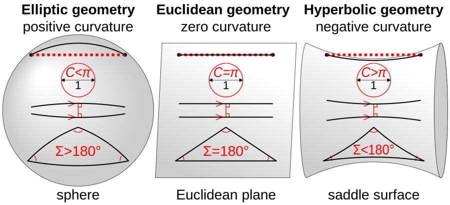
- Geometry is a branch of mathematics concerned with properties of space such as the shape, distance, size, and relative position of figures (points, lines, surfaces, solids).
- Geometry plays a critical role in various scientific disciplines, engineering, architecture, art, and everyday life.
- Traditionally geometry is identified with the Euclidean or Cartesian geometry
- However, there are various approaches for studing geometry and also various geometries.

Approaches

- Synthetic geometry: focuses on constructing proofs and understanding geometric properties and relationships through axioms and deductive reasoning.
- Analytic geometry: geometry using coordinate systems and algebraic methods.
- Projective geometry: studies properties preserved under projective transformations.
- Differential geometry: focuses on curves, surfaces, and smooth manifolds.
- Algebraic geometry: explores algebraic (polynomial) equations and their geometric solutions.
- Topology: Studies properties preserved under continuous deformations.

Classification based on parallelism

- Absolute (neutral) geometry: studies concepts independent of the notion of parallelism
- Playfair's axiom: Given a line l and a point $P \notin l$, how many lines l' through P parallel with L exist?
  - Euclidean geometry (a single parallel line)
  - Non-euclidean geometries
    - Elliptic geometry (no parallel lines)
    - Hyperbolic geometry (multiple parallel lines)



Classification based on transformation groups

Felix Klein introduced "Erlangen Program" aimed to categorize geometries based on transformation groups.

- Euclidean geometry (preserved by isometries: translations, rotations, reflections)
- Affine geometry (preserved by affine transformations: combinations of translations and linear transformations)
- Projective geometry (preserved by projective transformations: collineations)
- Hyperbolic geometry (preserved by hyperbolic transformations: e.g., disc preserving Möbius transformations)
- Elliptic geometry (preserved by elliptic transformations: e.g., sphere rotations)

Hyperbolic geometry

- In this talk I will focus on hyperbolic geometry, mostly on the analytic approach, with elements of projective geometry
- I will describe results on:
  - formalization of analytic hyperbolic geometry and its models: Poincaré disc and upper half-plane model, (Marić, Simić, Boutry)
  - formalization of gyrogroups and gyrovector spaces — a very convenient algebraic foundation of hyperbolic geometry, (Marić, in progress)
  - automated solving of construction problems in absolute and hyperbolic geometry (Marinković, Šukilović, Marić)
  - visualizations of hyperbolic geometry in JavaScript, (Marić, as a hobby)

Overview

Formalization of Complex plane geometry

- Filip Marić, Danijela Petrović: Formalizing Complex Plane Geometry. Annals of Mathematics and Artificial Intelligence, Springer, Volume 74, Issue 3, 2015.

Key insights

- It is very hard to formalize geometry if only geometric tools are available
- We must use results from other areas of mathematics
- Our formalizations are heavily based on:
  - complex numbers
  - linear algebra
  - algebra
  - ...

Complex numbers

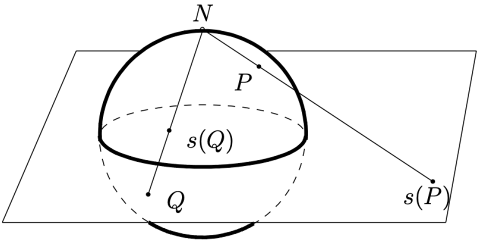
- Complex numbers are very convenient for formalizing geometry
- Formulas that use complex numbers are much simpler compared to formulas that use real numbers
- There are some very good books on this subject:
  - Tristan Needham: Visual Complex Analysis
  - Hans Schwerdtfeger: Geometry of Complex Numbers

Projective geometry and homogenous coordinates

- Considering degenerate cases is usually avoided by using projective geometry and homogenous coordinates
- Hyperbolic geometry is usually formalized in:
  - $\mathbb{R}P^2$ – real projective plane (convenient environment for the Klein/Beltrami model)
  - $\mathbb{C}P^1$ – complex projective line (convenient environment for the Poincaré disc model)
- Projective complex line $\mathbb{C}P^1$, also called extended complex plane is obtain when the complex plane $\mathbb{C}$ is extended by a single infinite point

Stereographic projection

- $\mathbb{C}P^1$ can be identified with a unit sphere by means of the stereographic projection.
- If points are projected from the north pole to the equatorial plane, then the north pole corresponds to the infinite point.



Complex projective line

- We have formalized $\mathbb{C}P^1$ and its geometry in Isabelle/HOL.
- A point is represented by homogenous coordinates:a pair (z_1, z_2) of complex numbers, not both equal to zero.
- If $z_2 \neq 0$, the pair represents a finite point $\frac{z_1}{z_1}$, and if $z_2 = 0$ represents the unique infinite point.
- Threfore, pairs (z_1, z_2) and (z_1', z_2') represent the same point iff they are proportional (for some nonzero complex factor k).
- Points are equivalence classes of the proportionality relation

Isabelle/HOL formalization

```
type_synonym cvec2 = "complex × complex"
typedef hc = "{v::cvec2.  v ≠ (0, 0)}"
definition eq_cvec2 ::  "cvec2 ⇒ cvec2 ⇒ bool" where
  "eq_cvec2 z1 z2 ⟺ (∃ k::complex.  k ≠ 0 ∧ z2 = k * z1)"
lift_definition eq_hc ::  "hc ⇒ hc ⇒ bool" is eq_cvec2
quotient_type cp1 = hc / eq_hc
```

Operations

- All four basic algebraic operations can be extended from $\mathbb{C}$ to $\mathbb{C}P^1$
- For example, addition can be performed in homogenous coordinates

$$(z_1, z_2) + (w_1, w_2) = (z_1 w_2 + z_2 w_1, z_2 w_2)$$

- Since points are equivalence classes it must be proved that this operation does not depend on the choice of representatives
- For finite points such operation agrees with ordinary addition in $\mathbb{C}$, and it holds that $\infty + z = z + \infty = \infty$ for all points $z \in \mathbb{C}P^1$

Circlines

- In addition to points, basic geometric objects are lines and circles
- $\mathbb{C}P^1$ enables their unified treatment, so generalized circles (called circlines) are considered.
- Circlines are given by quadratic equations (in complex homogenous coordinates):

$$Az_1\bar{z}_1 + Bz_1\bar{z}_2 + C\bar{z}_1z_2 + Dz_2\bar{z}_2 = 0$$

  for some real numbers A and D and complex numbers B and $C = \bar{B}$.
- Every circline is uniquely determined by its tree different points.
- A circline is a line iff it contains the infinite point (there is only one infinite point)

Stereographic projection

- If stereographic projection is applied, circlines in the extended complex plane (equatorial plane with the infinite point) correspond to circles on the unit sphere.
- Lines correspond to circles that contain the north pole.
- Each cicline can be identified by a plane in the (projective) 3d space

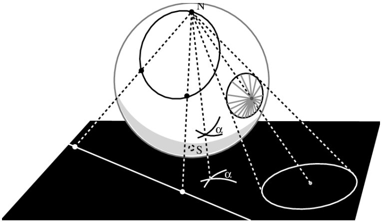


Figure author: Charles Delman

Matrix representation

- One of the key insights that facilitates formalization is that circlines should be represented by Hermitean matrices:

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

It holds $H^* = H$, giving that $\bar{A} = A$, $\bar{D} = D$, and $\bar{B} = C$.

- If the point vector is $z = (z_1, z_2)^t$, the circline equation becomes:

$$z^* \cdot H \cdot z = 0$$

- This is a quadratic form an it can be analyzed using standard methods of linear algebra

Matrix representation

- Proportional matrices (for some non-zero real factor k) yield same circlines
- Therefore, circlines are equivalence classes of Hermitean matrices under proportionality relation
- Orientation is preserved iff $k > 0$

Disc

- The sign of the quadratic form $z^* \cdot H \cdot z$ also determines interior and exterior.
- If H is a line, then both the exterior and interior are halfplanes.
- The disc equation is $z^* \cdot H \cdot z < 0$ (important for the Poincaré model that is given within the unit disc)

Möbius transformations

- Central objects in each projective geometry are those that preserve lines and incidence
- Fundamental transformations in $\mathbb{C}P^1$ are Möbius transformations that map circlines to circlines
- **Demo:** `mobius_mesh.html`

Möbius transformations

- In the complex plane $\mathbb{C}$ Möbius transformations are bilinear:

$$f(z) = \frac{az + b}{cz + d}$$

- In $\mathbb{C}P^1$ Möbius transforms are linear transformations, given by non-degenerate matrices acting on homogenous coordinates:

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \text{i.e., } z' = M \cdot z$$

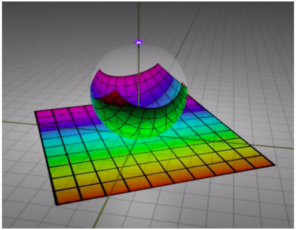
- Two matrices determine the same transformation iff they are proportional for some non-zero complex number k
- Möbius transformation is an equivalence class of non-degenerate complex matrices under the proportionality relation

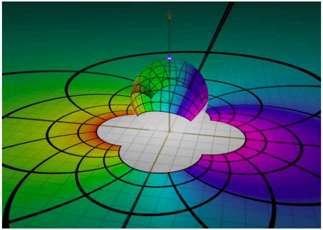
Möbius gruop

- Möbius transformations of $\mathbb{C}P^1$ form a group $PGL(2, \mathbb{C})$.
- Inverse transformation is represented by the inverse matrix, and composition of transformations is represented by matrix product
- Each Möbius transformation is a composition of translations, homotheties, rotations and inversions
- Möbius transformation is uniquely determined by images of three different points

Möbius transformations revealed

- Möbius transformations correspond to transformations of the unit sphere
- Youtube: Douglas Arnold and Jonathan Rogness, Möbius transformations revealed





Cross-ratio

- Cross-ratio transform given by w_1, w_2, w_3 is the unique Möbius transformation that maps $w_1 \mapsto 0$, $w_2 \mapsto 1$, and $w_3 \mapsto \infty$
- **Demo:** `cross_ratio.html`
- Cross-ratio of four points w_1, w_2, w_3, w_4 is the image of w_4 under the cross-ratio transform
- For finite points cross-ratio equals

$$\frac{(w_4 - w_1)(w_2 - w_3)}{(w_2 - w_1)(w_4 - w_3)}$$

- Möbius transformations preserve the cross-ratio of any 4 points
- 4 points lie on a circline iff their crossratio is real

Möbius transformations action on circlines

- Möbius transformations map circlines to circlines
- When the Möbius transfromation with the matrix M is applied to a circline H a circline

$$H' = (M^{-1})^* \cdot H \cdot (M^{-1})$$

  is obtained
- H and H' are congruent

Angles

- A very subtle notion:
  - Is the angle beween oriented or unoriented curves?
  - Is the angle itself oriented or unoriented?
  - Is the angle always convex or are non-convex angles allowed?
- Angle between circlines can be defined geometrically, as the angle between their tangents in the intersection point
- It can also be defined in purely algebraic terms

Algebraic definition of angle

- Assume that two circlines are given by the following Hermitean matrices:

$$H_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \quad H_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$$

- The cosine of the angle between them is given by:

$$\cos\alpha = \frac{-\Delta H_1 H_2}{2\sqrt{|H_1| \cdot |H_2|}}$$

- where

$$\Delta H_1 H_2 = A_1 D_2 - B_1 C_2 + A_2 D_1 - B_2 C_1$$

Angle preservation

- Möbius transformations are conformal i.e., they preserve angles
- It is easy to show that algebraically
- How can we be sure that what is formally proved corresponds to elementary geometric (non-algebraic) concepts?
- We formally prove that the traditional definition is equivalent to the algebraic one, which is used for proving properties

Overview

Formalization of the Poincaré disc model

- Danijela Simić, Filip Marić, Pierre Boutry Formalization of the Poincaré Disc Model of Hyperbolic Geometry, Journal of Automated Resoning, 2020.

H-points

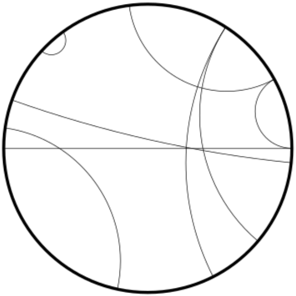
- We place the Poincaré model inside the interior of a unit disc within $\mathbb{C}P^1$.
- Unit disc is given by the equation $x^2 + y^2 < 1$ i.e., $||z||^2 = \bar{z}z < 1$, which is homogenized to $z_1\bar{z}_1 - z_2\bar{z}_2 = 0$ and represented by the matrix

$$H_{uc} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- H-points are elements z of $\mathbb{C}P^1$ such that $z^* \cdot H_{uc} \cdot z < 1$

H-lines

- Lines in the Poincaré model are circlines that are orthogonal to the unit circle



- The previous condition implies that h-lines are represented by Hermitean matrices of the form

$$H = \begin{pmatrix} A & B \\ \bar{B} & A \end{pmatrix}, |B|^2 > A^2$$

for a complex number B and a real number A

- Given two different h-points u and v, there is a unique h-line containing them, given by

$$\begin{aligned} A &= i \cdot (u\bar{v} - v\bar{u}) \\ B &= i \cdot (v(|u|^2 + 1) - u(|v|^2 + 1))) \end{aligned}$$

Isometries

- What transformations govern the Poincaré disc model?
- These are compositions of:
  - Möbius transformations that preserve the unit disc
  - Conjugation (reflection about the x-axis)
- These transformation form a group

Isometries — characterization

- It is formally shown that all Möbius transformations that preserve the unit disc are compositions of:
  - A rotation around the origin

    $$z \mapsto e^{i\theta} z$$

  - A Blaschke factor

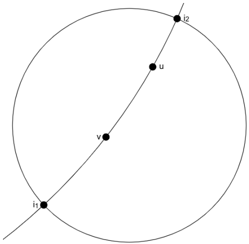
    $$z \mapsto \frac{z - a}{1 - \bar{a}z}$$

    a disc preserving "translation" that maps any given a in unit disc to the origin

Distance

- Poincaré disc is a metric space with the following h-distance
- **Demo:** `poincare_distance.html`
- Each h-line intersects the unit circle in two ideal points

$$\frac{B}{|B|^2}(-A \pm i \cdot \sqrt{|B|^2 - A^2})$$



Distance

- Let i_1 and i_2 be the ideal points of the h-line uv.
- Consider the cross-ratio function that maps i_1 to 0, i_2 to ∞ and u to 1, and its value for the point v — it is always a positive real number
- When v moves towards i_1, cross-ratio moves to 0
  When v moves to u, cross-ratio moves to 1
  When v moves to i_2, cross-ratio moves to ∞
- The logarithm moves from $-\infty$ to 0 to ∞
- The absolute value of the logarithm of the cross-ratio has all desired properties of a distance (triangle inequality, additivity for h-colinear points, ...)
- Therefore we define

$$d_h(u, v) = \left|\log \frac{(v - i_1)(u - i_2)}{(v - i_2)(u - i_1)}\right|$$

Distance

- It is proved that this formula reduces to

$$d(u, v) = \operatorname{arccosh}\left(1 + \frac{2 \cdot |u - v|^2}{(1 - |u|^2) \cdot (1 - |v|^2)}\right)$$

- This formula depends only on u and v (it does not include the ideal points)
- The distance function satisfies triangle inequality

$$d(u, v) \leq d(u, w) + d(w, u)$$

- Poincare disc with this distance function is a metric space
- Congruence of segments is reduced to distance equality

Distance preservation

- Isometries of the Poincaré disc preserve distances
- Easy to prove for conjugation and rotations, and a bit cumbersome for Blaschke factors

Circles

- h-circle is the set of h-points equidistant from a given h-point (considering h-distance)
- h-circle in the Poincaré disc is also a circle in the Euclidean sense (however, h-center and the Euclidean center are not the same)
- h-circle centered at u with h-radius r is Euclidean circle centered at

$$u_e = \frac{u}{(1 - |u|^2)\frac{\cosh r - 1}{2} + 1}$$

with radius

$$r_e = \frac{(1 - |u|^2)\sqrt{\frac{\cosh r - 1}{2} \cdot \frac{\cosh r + 1}{2}}}{(1 - |u|^2)\frac{\cosh r - 1}{2} + 1} = \frac{(1 - |u|^2)\sinh r}{(1 - |u|^2)(\cosh r - 1) + 2}$$

Betweenness

- One of the central geometric relations is betweenness of points
- How to define it?
- **Demo:** `between.html`
- A point v is between u and w if $v = u$ or $v = w$ or the crossratio of u, v, w and $\frac{1}{\bar{v}}$ is a negative real number
- Indeed, this cross-ratio is real iff v lies on the circline uw
- If u is mapped to 0 and w to ∞, then one of the two arcs uv contains only the positive values of the cross-ratio and the other only the negative values
- The point $\frac{1}{\bar{v}}$ is the inversion of v and it is outside the unit circle. If the cross-ratio maps it to 1, then the arc that is contained within the unit disc yields negative cross-ratio values

Cross-ratio definitions

- Both the distance and betweenness are defined using cross-ratio
- This makes it easier to prove their properties
- Since Möbius transformations preserve cross-ratio, they also preserve distances and betweenness
- This enables wlog reasoning when analyzing properties of distance and betweenness

Tarski axioms

- Finally, we proved that the Poincaré disc satisfies all Tarski's axioms of geometry (and the negation of the parallels postulate)
- Hardest problems were the ones that required finding intersections of circlines
- Wlog reasoning came to the rescue
- Möbius transformations were employed to map one circline to the x-axis, since we derived a relatively simple expression for the intersection of a circline and the x-axis

$$x = \frac{-\mathrm{Re}\ B}{A} + \frac{\mathrm{sgn}(\mathrm{Re}\ B) \cdot \sqrt{(\mathrm{Re}\ B)^2 - A^2}}{A}$$

- The hardest axiom to prove was the Pasch axiom

Overview

Vectors

- Euclidean geometry gives us a very natural concept of a vector
- Vector addition is associative and commutative, so vectors form an Abellian group under addition
- Vectors are naturally multiplied by a scalar, giving rise to a vector space
- Dot (inner) product and vector norm are also easily defined and they give rise to the Euclidean metric ($d(A, B) = ||\overrightarrow{AB}||$)
- As we have seen, standard expositions of hyperbolic geometry do not use vectors!

Gyrogoups and gyrovector spaces

- Inspired by physics, Abraham A. Ungarn introduced a framework that unifies special relativity theory and hyperbolic geometry
- The approach is based on gyrovectors — a generalization of vectors appropriate for hyperbolic geometry
- Once the appropriate algebraic foundation is introduced, fascinating analogies between Euclidean and hyperbolic goemetry are revealed

Classical vs. Relativistic physics

- Classical, Newtonian kinematics takes place in the Euclidean space
- Velocities are represented by vectors and classic (Galilei) velocity adition is the ordinary vector addition, that is both commutative and associative
- Relativistic (Einstein) velocity addition of admissible velocities is non-commutative and non-associative
- Minkowski laid down theoretical foundations of special relativity, but did not emphasize its connections with hyperbolic geometry
- Ungarn expored the idea that hyperbolic geometry governs velocities in relativistic physics in the same way that Euclidean geometry governs velocities in prerelativistic physics

Physics and hyperbolic geometry

- Let us consider two seemingly unrelatead examples:
  - Disc preserving Möbius transformations
  - Einstein's velocity addition from special relativity theory
- Both are bound to the interior of a unit disc (sphere), since all relativistically admisiblle velocities are bound by the speed of light c ($||v|| < c$).

Möbius gyrogroup

- All disc preserving Möbius transformations are compositions of translations (Blaschke factors) and rotations:

$$z \mapsto e^{i\theta}\frac{z+a}{1+\bar{a}z}$$

- Inspired by this, let $\oplus$ denote Möbius addition of "vectors" (points in the complex unit disc):

$$u \oplus_M v = \frac{u+v}{1+\bar{u}v}$$

- Disc preserving transfroms are then expressed as:

$$z \mapsto e^{i\theta}(a \oplus_M z)$$

Möbius gyrogroup

- If the center O, u and v are h-collinear Möbius addition $\oplus_M$ is commutative and associative.
- In the general case it is neither commutative nor associative.
- The operation $\oplus_M$ does not yield a group. Whan algebraic laws does it satisfy?

Gyrogroups

- **Demo:** `gyro_mobius_addition.html`
- $u \oplus_M v$ and $v \oplus_M u$ are not the same, but they have the same norm, so they are linked by a rotation: there is a unique rotation that takes $v \oplus_M u$ to $u \oplus_M v$. Denote that rotation by

$$\mathrm{gyr}_M[u, v] = \frac{1 + \bar{u}v}{1 + \bar{v}u}$$

  It is called Möbius gyration of u and v.
- Möbius gyration repairs commutativity:

$$u \oplus_M v = \mathrm{gyr}_M[u, v](v \oplus_M u)$$

Associativity

- (Not) surprisingly, gyration also repairs associativity:

$$u \oplus_M (v \oplus_M w) = (u \oplus_M v) \oplus_M \mathrm{gyr}_M[u, v]w$$

$$(u \oplus_M v) \oplus_M w = u \oplus_M (v \oplus_M \mathrm{gyr}_M[v, u]w)$$

Inverses

- What about inverses?
- They are used to solve equations like $x \oplus_M u = v$.
- Denote $\ominus_M u = -u$ and $u \ominus_M v = u \oplus_M (\ominus_M v)$.

$$\begin{aligned} x &= x \oplus_M 0 \\ &= x \oplus_M (u \ominus_M u) \\ &= (x \oplus_M u) \oplus_M \mathrm{gyr}_M[x, u](\ominus_M u) \\ &= (x \oplus_M u) \ominus_M \mathrm{gyr}_M[x, u]u \\ &= v \ominus_M \mathrm{gyr}_M[x, u]u \end{aligned}$$

- How to eliminate x from the RHS?

Inverses

- Loop property of gyration:

$$\begin{aligned} \mathrm{gyr}_M[u \oplus_M v, v] &= \mathrm{gyr}_M[u, v] \\ \mathrm{gyr}_M[u, v \oplus_M u] &= \mathrm{gyr}_M[u, v] \end{aligned}$$

- Continuing the previous calculation we get

$$\begin{aligned} x &= v \ominus_M \mathrm{gyr}_M[x, u]u \\ &= v \ominus_M \mathrm{gyr}_M[x \oplus_M u, u]u \\ &= v \ominus_M \mathrm{gyr}_M[v, u]u \end{aligned}$$

- Therefore gyration also repairs inverses:

$$x \oplus_M u = v \Leftrightarrow x = v \ominus_M \mathrm{gyr}_M[x, v]u$$

- We almost have a group-like structure.

Einstein velocity addition

- Consider now the next example.
- In one of his 1905. *(annus mirabilis)* papers, Einistein introduced relativistic addition law for relativistically admissible velocities. Let $u, v \in \{w \in \mathbb{R}^3 : ||w|| < c\}$.

$$u \oplus_E v = \frac{1}{1 + \frac{u \cdot v}{c^2}} \left(u + \frac{1}{\gamma_u} v + \frac{1}{c^2} \frac{\gamma_u}{1 + \gamma_u} (u \cdot v) u \right)$$

$$\gamma_u = \frac{1}{\sqrt{1 - \frac{||u||^2}{c^2}}}$$

Einstein's gyrogroup

- What algebraic properties does the Einstein addition $\oplus_E$ poses?
- As for $\oplus_M$, adding parallel velocities is both commutative and associative, but the general case is neither.
- As for $\oplus_M$, $u \oplus_E v$ and $v \oplus_E u$ are connected by a rotation (called the Thomas precession in special relativity, and experimentally determined)
- Thomas precession plays the role of gyration for Einstein addition
- Once gyrations are introduced, algebraic properites of Möbius and Einstein addition become very similar. They both give rise to algebraic structure called gyrogroup.

Gyrogrups

- Properties of Gyrogroups can be specified axiomatically. A groupoid $(G, \oplus)$ is a gyrogroup if following axioms hold.
  1. There is an element $0 \in G$ such that for all $a \in G$ it holds $0 \oplus a = a$ (left identity)
  2. For each $a \in G$ there is $\ominus a \in G$ such that $\ominus a \oplus a = 0$ (left inverse)
  3. For each $a, b, z \in G$, there exists a unique $\mathrm{gyr}[a, b]z \in G$ such that $a \oplus (b \oplus z) = (a \oplus b) \oplus (\mathrm{gyr}[a, b]z)$ (left gyroassociativity)
  4. The map $\mathrm{gyr}[a, b]$ maps each z to $\mathrm{gyr}[a, b]z$. For each $a, b \in G$, $\mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)$ (gyroautomorphism)
  5. $\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$ (left loop)

Gyrocommutativity, duals

- A gyrogroup is gyrocommutative if $a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$
- Gyration can always be expressed in terms of addition:

$$\mathrm{gyr}[a, b]z = \ominus(a \oplus b) \oplus (a \oplus (b \oplus z))$$

- Each gyrogroup gives rise to a dual operation called gyrocooperation, defined by: $a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b$
- Cooperation gives rise to some nice symmetries. For example:

$$x \oplus a = b \Leftrightarrow x = b \boxminus a$$

Scalar multiplication

- By deriving formula for $u_1 \oplus_E u_2 \oplus_E \ldots \oplus_E u_k$ and generalizing it, we define:

$$\begin{aligned} t \otimes_E v &= c\frac{(1+\frac{||v||}{c})^t - (1-\frac{||v||}{c})^t}{(1+\frac{||v||}{c})^t + (1-\frac{||v||}{c})^t}\frac{v}{||v||} \\ &= c \tanh\left(r \cdot \operatorname{arctanh}\frac{||v||}{c}\right)\frac{v}{||v||} \end{aligned}$$

- Exactly the same definition applies for the Möbius scalar multiplication $\otimes_M$.

Gyrovector space

- Einstein and Möbius scalar multiplication pose properties similar (but not exactly the same) to vector spaces
- Their algebraic properties are exactly captured by the notion of gyrovector spaces, given axiomatically.
- Some axioms:
  - $n \otimes v = \underbrace{v \oplus \ldots \oplus v}_{n \text{ times}}$
  - $(t_1 + t_2) \otimes v = t_1 \otimes v \oplus t_2 \otimes v$
  - $(t_1 \cdot t_2) \otimes v = t_1 \otimes (t_2 \otimes v)$
  - ...
- However:

$$t \otimes (u \oplus v) \neq t \otimes u \oplus t \otimes v$$

Distance function

- Definition of distance is inspired by the Euclidean one:

$$d(u, v) = ||u \ominus v||$$

- It satisfies gyrotriangle inequality:

$$d(a, c) \leq d(a, b) \oplus d(b, c)$$

- Finding inverse hyperbolic tangent gives us usual hyperbolic metrics that satisfy the ordinary triangle inequalities.

$$h(a, b) = \text{arctanh } d(a, b)$$

Hyperbolic metrics

- Einstein addition gives rise to the Klein-Beltrami metric
- Möbius addition gives rise to the Poincaré disc metric

Isomorphism

- Klein-Beltrami and Poincaré disc are isomorphic, with very simple isomorphisms:

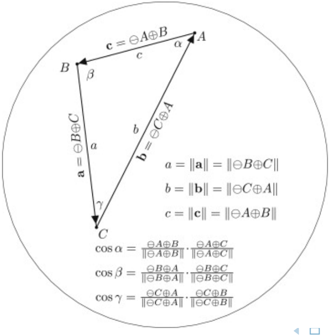
$$a_E = 2 \otimes a_M, \quad a_M = \frac{1}{2} \otimes a_E$$

- Möbius and Einstein addition are easilty expressible one from another:

$$\begin{aligned} u \oplus_M v &= \frac{1}{2} \otimes ((2 \otimes u) \oplus_E (2 \otimes v)) \\ u \oplus_E v &= 2 \otimes ((\frac{1}{2} \otimes u) \oplus_M (\frac{1}{2} \otimes v)) \end{aligned}$$

Vectors and angles

- Many notions can be defined in analogy with the Euclidean geometry.
- Vector between two points is the "difference" of those two points.
- Cosine of the angle between two vectors is the "scalar product" of normalized vectors.



Geodesics

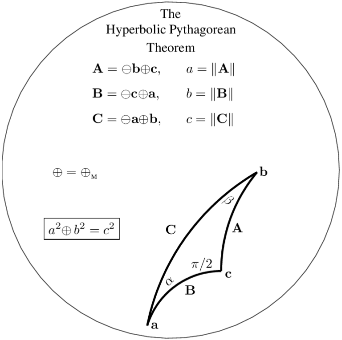
- Definition of (geodesic) lines is inspired by the Euclidean case and paremetric equation of line $a + t \cdot (b - a)$, $t \in \mathbb{R}$.
- Line trough points a and b is given by:

$$a \oplus t \otimes (\ominus a \oplus b), \quad t \in \mathbb{R}$$

- **Demo:** `mobius_geodesic.html`, `einstein_geodesic.html`
- For Möbius addition geodesics are circle segments orthogonal to the disc, giving Poincaré disc model
- For Einstein addition geodesics are chords of the disc, giving Klein-Beltrami disc model

Pythagorean theorem

- In the Poincaré disc model, Pythagorean theorem holds, but is expressed unusually: $\cosh a \cdot \cosh b = \cosh c$
- Using gyrovectors Pythagorean theorem takes its classic form $a^2 \oplus b^2 = c^2$:



Constructions

- Vesna Marinković, Tijana Šukilović, Filip Marić, On automating triangle constructions in absolute and hyperbolic geometry, ADG 2021.

Goal

- Many ruler and compass constructions are valid only in Euclidean geometry
- We want to automatically find constructions that are valid in absolute geometry
- We want to automatically find constructions that are valid in hyperbolic geometry

Definitions and pseduo-elements

- In the Euclidean case many notions can be defined in equivalent ways. For example,
  - a median is the segment that connect a triangle vertex with the midpoint of its opposite side
  - a median is a segment that divides the triangle area in two exact halves
- In hyperbolic case these need not coincide, so we define different objects For example, we distinguish:
  - median (definition 1) and
  - pseudo-median (definition 2)
- Some Euclidean theorems hold only for pseudo-elements (e.g., Euler line does not exist, but pseudo-Euler line exists)
- Unfortunately, some pseudo-elements are not ruler and compass constructible

Theorems of absolute geometry (weaker than in Euclidean geometry)

- The three medians of a triangle intersect in one point (the centroid G)
- The three internal angle bisectors of a triangle intersect in one point (the incenter I)
- The perpendicular bisectors of triangle sides belong to the same pencil of lines (the circumcenter need not exist)
- The altitudes a triangle belong to the same pencil of lines (the orthocenter need not exist)
- ...

Euclidean lemmas that fail in hyperbolic geometry

- The centroid G does not divide the median in 2:1 ratio
- The inscribed angle subtended by a diameter need not be right
- Locus of points subtending a segment under a given angle is not a circular arc
- Equidistant curve is not a line
- . . .

ArgoTriCS in hyperbolic geometry

- We have identified definitions, lemmas and primitive constructions relevant for absolute and hyperbolic geometry
- We have adapted ArgoTriCS for solving constructions in absolute and hyperbolic geometry by providing it with appropriate lemmas and construction steps
- Hyperbolic triangle has more "significant points" than the Euclidean triangle (in the Euclidean case many points coincide)
- Loci in hyperbolic geometry can be more complicated than in the Euclidean case where many loci are circles and lines
- Ruler and compass constructions are much harder in absolute and hyperbolic geometry (we believe that many problems are not RC-constructible)

ArgoDG

- ArgoDG – a lightweight, open-source JavaScript library for dynamic visualization
- Some users (like myself) prefer typing to clicking
- A library (API) can be better than a dedicated language

Demo

DEMO

Conclusions

- We have described two different analytical approaches for formalizing hyperbolic geometry
- Both require advanced algebraic machinery (linear algebra over complex field, or non-commutative, non-associative algebraic structures)
- Formalization would be extremely hard (practically impossible) if "Without loss of generality" is not used
- Euclidean geometry gives us polynomial equations over classic fields, and automated reasoning reduces to classic algorithms over polynomials (e.g., Gröbner basis)
- In hyperbolic geometry usually we deal with expressions that involve transcendental functions (e.g., sinh, cosh, . . .)
- Gyrovectors give expressions identical to Euclidean, and polynomial equations, but they hide non-standard operations, whose theory is not well-developed