Automated Completion of Statements and Proofs in Synthetic Geometry: an Approach based on Constraint Solving

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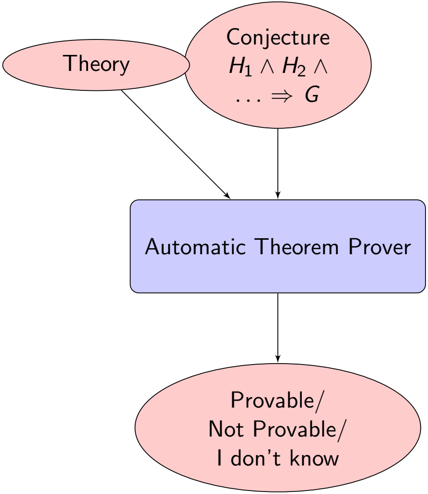
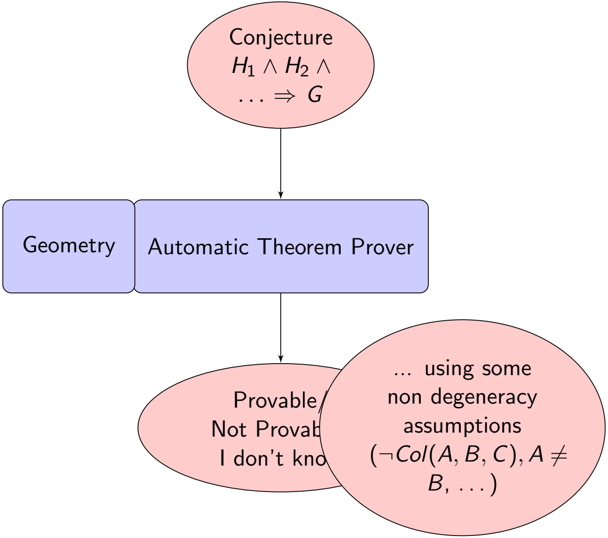
Automated deduction in general



Automated deduction in geometry

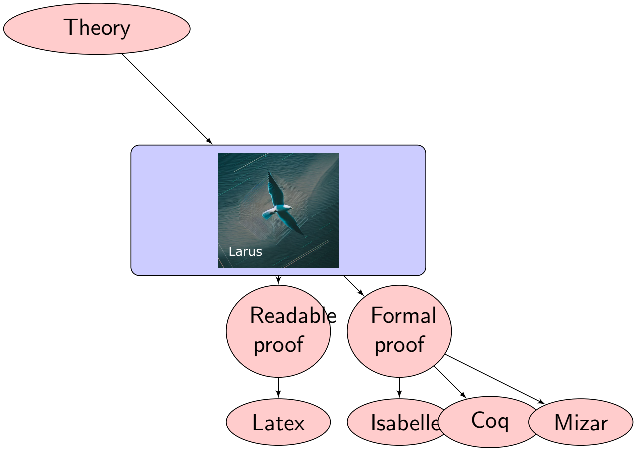


Conjecturing and proving

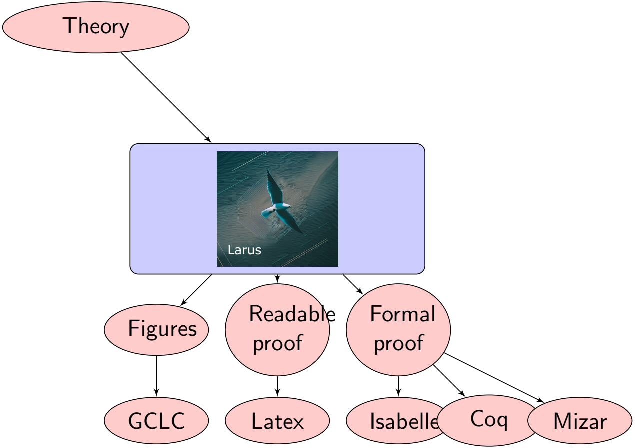
- But real mathematical activity does not fit into this picture.
- Conjecturing/refuting/proving/producing lemmas, theories or definitions are interlaced activities. See Lakatos [1]

[1] *Proofs and Refutations* (1976).

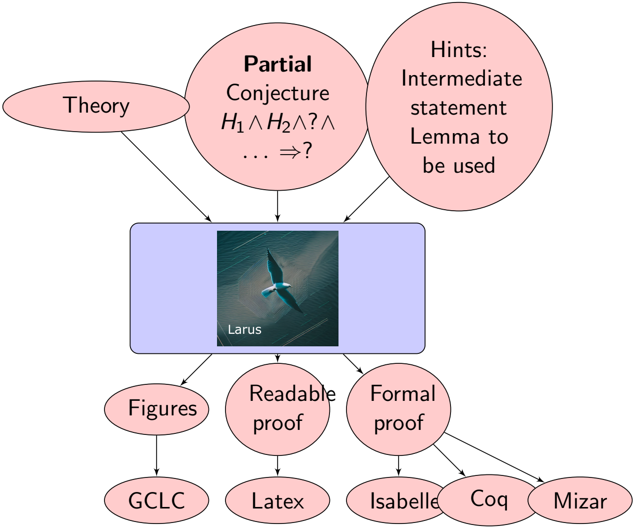
Our approach



Our approach



Our approach



Coherent Logic / Finitary Geometric Implications

- A formula of coherent logic (universally closed):

$$A_0(\vec{x}) \wedge \ldots \wedge A_{n-1}(\vec{x}) \Rightarrow \exists \vec{y}(B_0(\vec{x}, \vec{y}) \vee \ldots \vee \; B_{m-1}(\vec{x}, \vec{y}))$$

  where universal closure is assumed, A_i denotes an atomic formula, and B_j denotes a conjunction of atomic formulae.
- No function symbols of arity > 0 and no negations
- Many theories can be simply formulated in CL
- Every FOL theory can be translated into CL, possible with additional predicate symbols
- For instance, for each predicate symbol R, a new symbol $\overline{R}$ is introduced for $\neg R$, and the axioms: $\forall \vec{x}(R(\vec{x}) \wedge \overline{R}(\vec{x}) \Rightarrow \bot)$, $\forall \vec{x}(R(\vec{x}) \vee \overline{R}(\vec{x}))$

Inference System for Coherent Logic

$$\frac{\Gamma, ax, A_0(\vec{a}), \ldots, A_{n-1}(\vec{a}), \underline{B_0(\vec{a}, \vec{b}) \vee \ldots \vee B_{m-1}(\vec{a}, \vec{b})} \vdash P}{\Gamma, ax, A_0(\vec{a}), \ldots, A_{n-1}(\vec{a}) \vdash P} \text{ MP}$$

where ax is
$A_0(\vec{x}) \wedge \ldots \wedge A_{n-1}(\vec{x}) \Rightarrow \exists \vec{y}(B_0(\vec{x}, \vec{y}) \vee \ldots \vee B_{m-1}(\vec{x}, \vec{y}))$

$$\frac{\Gamma, \underline{B_0(\vec{c})} \vdash P \quad \ldots \quad \Gamma, \underline{B_{m-1}(\vec{c})} \vdash P}{\Gamma, B_0(\vec{c}) \vee \ldots \vee B_{m-1}(\vec{c}) \vdash P} \text{ QEDcs (case split)}$$

$$\frac{}{\Gamma, \underline{B_i(\vec{a}, \vec{b})} \vdash \exists \vec{y}(B_0(\vec{a}, \vec{y}) \vee \ldots \vee B_{m-1}(\vec{a}, \vec{y}))} \text{ QEDas (assumption)}$$

$$\frac{}{\Gamma, \bot \vdash P} \text{ QEDefq (ex falso quodlibet)}$$

Inference System for Coherent Logic: Example

Consider the following two axioms:

ax1: $\forall x\ (p(x) \Rightarrow r(x) \vee q(x))$ ax2: $\forall x\ (q(x) \Rightarrow \bot)$

and the conjecture: $\forall x\ (p(x) \Rightarrow r(x))$

$$\dfrac{\dfrac{\dfrac{}{ax1, ax2, p(a), r(a) \vdash r(a)}\ \text{QEDas} \qquad \dfrac{\dfrac{}{ax1, ax2, p(a), q(a), \bot \vdash r(a)}\ \text{QEDefq}}{ax1, ax2, p(a), q(a) \vdash r(a)}\ \text{MP(ax2)}}{ax1, ax2, p(a), r(a) \vee q(a) \vdash r(a)}\ \text{QEDcs}}{ax1, ax2, p(a) \vdash r(a)}\ \text{MP(ax1)}$$

The same proof in a forward manner, in a natural language form:

Consider an arbitrary a such that: $p(a)$. It should be proved that $r(a)$.

1. $r(a) \vee q(a)$ (by MP, from $p(a)$ using axiom ax1; instantiation: $X \mapsto a$)
    2. Case $r(a)$:
        3. Proved by assumption! (by QEDas)
    4. Case $q(a)$:
        5. $\bot$ (by MP, from $q(a)$ using axiom ax2; instantiation: $X \mapsto a$)
        6. Contradiction! (by QEDefq)
7. Proved by case split! (by QEDcs, by $r(a), q(a)$)

Starting Ideas

- The pure forward chaining approach to ATP does not take the goal into account.
- SAT/SMT solvers have seen huge progress in the recent years.
- Encoding the problem of finding a Coherent Logic proof into SAT/SMT theories can give a form of multidirectional reasoning.

Theorem Proving as Constraint Solving

- In traditional automated proving:
  - the search is performed over a set of formulae, and it terminates once the goal formula or contradiction is found.
  - a proof can then be reconstructed as a byproduct of this process.
- In our approach, *proving as constraint solving*:
  - a proof of a given formula can be represented by a sequence of natural numbers, meeting some constraints;
  - the search is performed globally over a set of possible proofs (i.e., over a set of possible sequences of natural numbers);
  - a proof is found by a solver that finds a sequence that meets these conditions.
  - a proper proof can be reconstructed from the found sequence.

Encoded Proof: Example

```
0.   1  0 0    2 0      /* Nesting: 1; Step kind:0 = Assumption;
                           Branching: no; p2(a) */
1.   1 13 1    4 0 6 0 /* Nesting: 1; Step kind:13 = MP-axiom:13;
                           Branching: yes; p4(a) or p6(a) */
                     0 /* From steps: (0) */
                     0 /* Instantiation */
2.   2  2 0    4 0      /* Nesting: 2; Step kind:2 = First case;
                           Branching: no; p4(a) */
3.   2 10               /* Nesting: 2; Step kind:10 =
                                            QED by assumption; */
4.   3  3 0    6 0      /* Nesting: 3; Step kind:3 = Second case;
                           Branching: no; p6(a) */
5.   3 14 0    0        /* Nesting: 3; Step kind:14=MP-axiom:14);
                           Branching: no; p0() */
                     4 /* From steps: (4) */
                     0 /* Instantiation */
6.   3 11               /* Nesting: 3; Step kind:11 = QED by EFQ;*/
7.   1  9               /* Nesting: 1; Step kind:9 = QED by cases;*/
```

Related work

Surprisingly (as far as we know), this approach has hardly been studied extensively. Only, partly related:

- Todd Deshane, Wenjin Hu, Patty Jablonski, Hai Lin, Christopher Lynch, and Ralph Eric McGregor. *Encoding First Order Proofs in SAT*, CADE-21, 2007.
- Jeremy Bongio, Cyrus Katrak, Hai Lin, Christopher Lynch, and Ralph Eric McGregor. *Encoding First Order Proofs in SMT*. ENTCS, 198(2):71–84, 2008.

Proof encoding and constraints

- We generate constraints that a sequence of natural numbers represents a valid proof.
- Proofs by cases are encoded by associating `nesting` information to each proof step.
- Each proof consists of steps of the following types: `Assumption`, `MP`, `FirstCase`, `SecondCase`, `QEDbyCases`, `QEDbyAssumption`, `QEDbyEFQ`
- `Contents` corresponds to a disjunction in a proof step, `Goal` is *true* iff `Contents` is the goal...
- There are also global constraints

Example: Constraints for steps QEDbyEFQ

- Each proof step has one of the above sorts and meets some constraints
- For instance, if the step s is of the kind QEDbyEFQ, then the following conditions hold:

1. $\mathrm{StepKind}(s) = \mathrm{QEDbyEFQ}$
2. $s > 0$
3. $\mathrm{contents}(s-1)(0) = \bot$
4. step s is the goal
5. $\mathrm{Nesting}(s) = \mathrm{Nesting}(s-1)$

Pipeline

1. A maximal proof length M is given.
2. Proof steps and the constraints are encoded by natural numbers.
3. A constraint solver (for linear arithmetic, for instance), is invoked to find a model.
4. There is a proof of length $\leq M$ iff there is a model for the constraints.
5. If there is a model, then a proof can be reconstructed from it.
6. A proof for a proof assistant, a readable proof, an illustrated proof then can be constructed from the proof.

Completing incomplete premises: Abduction (1/5)

Given a theory $\mathcal{T}$ and a conjecture G, assuming that $\mathcal{T} \not\models G$ and $\mathcal{T} \not\models \neg G$, the objective is to find a set of atomic formulae F, such that it holds:

- $\mathcal{T}, F \vdash G$
- the set $\{\mathcal{T}, F\}$ is consistent

The formulas in F are called the abducts.

Completing incomplete premises: Abduction (2/5)

There can be additional conditions. In Larus (an abduct makes the step i):

1. StepKind(i) =Assumption
2. Nesting(i) $= 1$
3. Cases(i) $=$ *false*
4. ContentsPredicate($i, 0$) $<$ *sizeof*(*Signature*)
5. for each argument j (up to maximal arity): ContentsArgument($i, 0, j$) $<$ *sizeof*(*Constants*)
6. Goal(i) $=$ *false*
7. ContentsPredicate(i,0) $\neq \perp$

1. ax0 : $\forall X\ (p(X) \Rightarrow q(X))$
2. ax1 : $\forall X\ (q(X) \Rightarrow r(X) \vee s(X))$
3. ax2 : $\forall X\ (r(X) \Rightarrow \bot)$

Conjecture: $\forall X\ (s(X))$

The conjecture cannot be proved, but Larus offers two abducts:

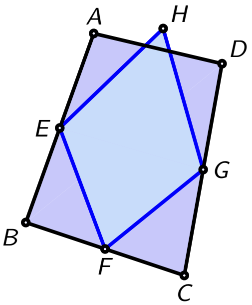
```
1. set:
  1. ((q(b)))
Abducts CONSISTENT!
```

Conjecture: $\forall X\ (\ q(X) \Rightarrow\ s(X))$

```
2. set:
  1. ((p(b)))
Abducts CONSISTENT!
```

Conjecture: $\forall X\ (\ p(X) \Rightarrow\ s(X))$

Completing incomplete premises: Abduction (4/5)



Let ABCD be a quadrilateral. Let E, F, G the midpoints of AB, BC et CD respectively. Let H be a point.
Under which assumption the quadrilateral EFGH is a parallelogram ?

$\longrightarrow$ H should be the midpoint of segment AD.

Consider arbitrary a, b, c, d, e, f, g, h such that:

- $\neg col(b, d, a)$,
- $\neg col(b, d, c)$,
- $\neg col(a, c, b)$,
- $\neg col(a, c, d)$,
- $\neg col(e, f, g)$,
- $b \neq d$,
- $a \neq c$,
- $midpoint(a, e, b)$,
- $midpoint(b, f, c)$,
- $midpoint(c, g, d)$.

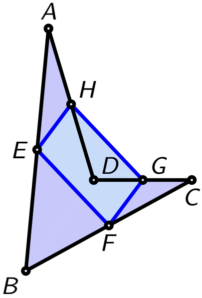
It should be proved that $pG(e, f, g, h)$.

Abducts found:

- $midpoint(d, h, a)$

1. $par(a, c, e, f)$ (by MP, from $\neg col(a, c, b)$, $midpoint(b, f, c)$, $midpoint(a, e, b)$ using axiom triangle_mid_par_strict; instantiation: $A \mapsto a$, $B \mapsto c$, $C \mapsto b$, $P \mapsto f$, $Q \mapsto e$)

...



Let ABCD be a quadrilateral. Let E, F, G, H be the midpoints of the segments [AB], [BC], [CD] and [DA] respectively.
What can we say about EFGH ?

$\longrightarrow$ EFGH is a parallelogram.

Completing incomplete goals: Deducts (2/2)

It should be proved that $\_(e, f, g, h)$.

2. $par(a, c, e, f)$ (by MP, from $\neg col(a, c, b)$, $midpoint(b, f, c)$, $midpoint(a, e, b)$ using axiom triangle_mid_par_strict; instantiation: $A \mapsto a$, $B \mapsto c$, $C \mapsto b$, $P \mapsto f$, $Q \mapsto e$)
3. $par(a, c, h, g)$ (by MP, from $\neg col(a, c, d)$, $midpoint(c, g, d)$, $midpoint(a, h, d)$ using axiom triangle_mid_par_strict; instantiation: $A \mapsto a$, $B \mapsto c$, $C \mapsto d$, $P \mapsto g$, $Q \mapsto h$)
4. $par(e, f, g, h)$ (by MP, from $par(a, c, e, f)$, $par(a, c, h, g)$, $\neg col(e, f, g)$ using axiom lemma_par_trans; instantiation: $A \mapsto e$, $B \mapsto f$, $C \mapsto a$, $D \mapsto c$, $E \mapsto g$, $F \mapsto h$)
5. Proved by assumption! (by QEDas)

Completing incomplete proofs: Hints (1/3)

9.3 Lemma. $BaAc \wedge m\in A \wedge Mamc \wedge r\in A \to \forall b[a \underset{r}{\approx} b \to BbAc]$.

(Wenn a und c auf entgegengesetzten Seiten der Geraden A liegen, und zwar spiegelbildlich bezüglich eines Punktes von A, und r auf A liegt, so liegt jeder Punkt b der Halbgeraden H(ra) entgegengesetzt zu c bezüglich A, Abb. 33.)

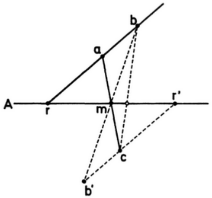


Abb. 33

Beweis: Sei $a \underset{r}{\approx} b$. Nach Def. 6.1(ii) ist $Brba \vee Brab$.

Completing incomplete proofs: Hints (2/3)

- *Hint* can be found in an informal proof (for instance, in a textbook), from machine verifiable proof, or from memory!
- For a proof or a proof step, hint can specify:
  - the predicate symbol
  - arguments in the atomic formula
  - the ordinal of a proof step
  - the axiom applied in the step
  - ...
- In other provers, such hints are extremely difficult to use
- In some cases, hints can lead to significant speed-ups

Completing incomplete proofs: Hints (3/3)

- Using this approach, the user can add constraints either to help the prover or to find a specific proof.
- Examples:
  - predicate r must appear somewhere in the proof:
    `fof(hintname0, hint, r(?,?), _, _)`
  - ax2 must be used in the proof at step 3, instantiating both arguments with the same value
    `fof(hintname0, hint, _, 3, ax2(A,A))`

Future work

- Many generated abducts/deducts are ,,uninteresting“ or mutually similar
- There are different restrictions in abduction considered in the literature and we will consider different criteria for filtering out ,,interesting“ abducts/deducts (for instance, minimal in some sense)

Conclusions

- We have shown that we can extend a prover, which uses constraint solving, so that it can complete:
  - partially specified hypotheses
  - partially specified conclusions
  - partially specified proofs
- All three tasks fit naturally into *proving as constraint solving* paradigm: it is only that some constraints are added or deleted
- To our knowledge, this approach is new, and we are not aware of any other systems that tackle these three completion problems.