Towards an independent version of Tarski's system of geometry

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ADG 2023
Belgrade, September 21, 2023

Introduction

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- Hilbert's *Grundlagen der Geometrie* contains a chapter dedicated to independence properties.
- Tarski's System of Geometry and the problem of its independence was carefully studied by Gupta.

Outline

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- 11 axioms.
- A parameter controls the dimension.
- Good meta-theoritical properties.

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Pasch's axiom

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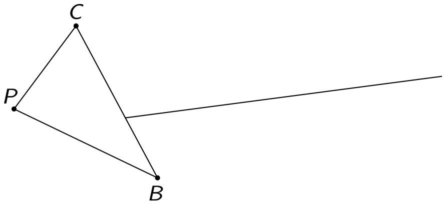
Axiom (Pasch)

$$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$$



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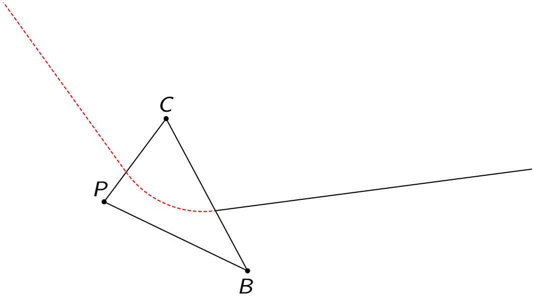

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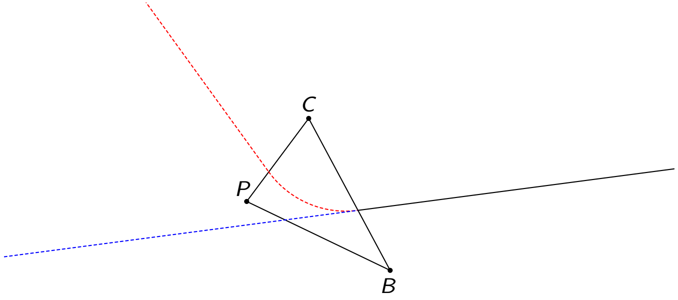

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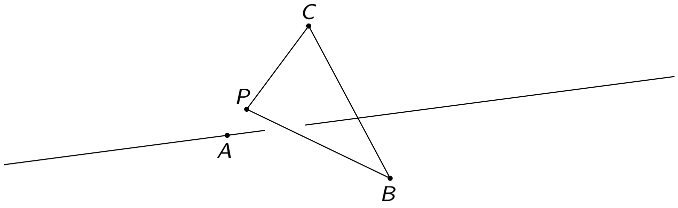

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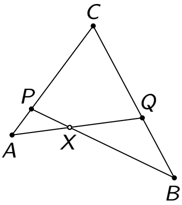

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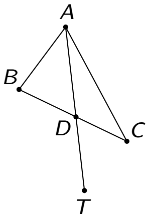

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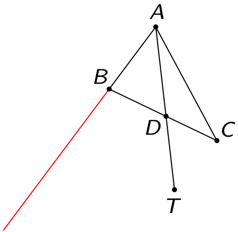

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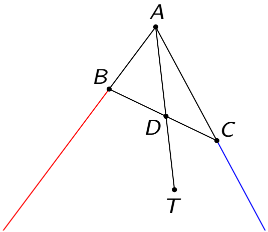

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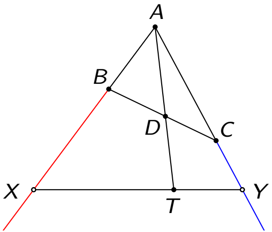

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Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A—B—E \wedge BE \equiv CD$
Pasch	$A—P—C \wedge B—Q—C \Rightarrow \exists X, P—X—B \wedge Q—X—A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A—B—C \wedge A'—B'—C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A—B—C \vee B—C—A \vee C—A—B$
Euclid	$A—D—T \wedge B—D—C \wedge A \neq D \Rightarrow$ $\exists XY, A—B—X \wedge A—C—Y \wedge X—T—Y$
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Point equality decidability	$X = Y \vee X \neq Y$

Overview of the formalization

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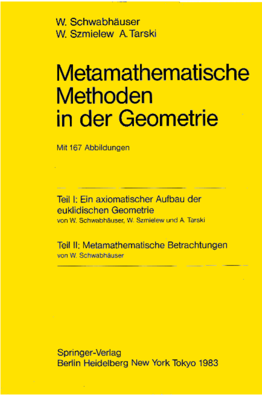
W. Schwabhäuser
W. Szmielew A. Tarski

Metamathematische Methoden in der Geometrie

Mit 167 Abbildungen

Teil I: Ein axiomatischer Aufbau der euklidischen Geometrie
von W. Schwabhäuser, W. Szmielew und A. Tarski

Teil II: Metamathematische Betrachtungen
von W. Schwabhäuser

Springer-Verlag
Berlin Heidelberg New York Tokyo 1983

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- $A–B–C := \exists k, 0 \leq k \leq 1 \wedge B - A = k(C - A)$.
- We mechanized the proof that all the axioms, excluding continuity, hold in *this* model.

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Pasch	$A—P—C \wedge B—Q—C \Rightarrow \exists X, P—X—B \wedge Q—X—A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A—B—C \wedge A'—B'—C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A—B—C \vee B—C—A \vee C—A—B$
Euclid	$A—D—T \wedge B—D—C \wedge A \neq D \Rightarrow$ $\exists XY, A—B—X \wedge A—C—Y \wedge X—T—Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A—X—Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow$ $X—B—Y \vee X = B \vee B = Y)$

Gupta's axiom system

Transitivity for betweenness	$A{-}B{-}D \wedge B{-}C{-}D \Rightarrow A{-}B{-}C$
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Lower 2-Dimensional	$\exists ABC, \neg A{-}B{-}C \wedge \neg B{-}C{-}A \wedge \neg C{-}A{-}B$
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Euclid	$A{-}D{-}T \wedge B{-}D{-}C \wedge A \neq D \Rightarrow$ $\exists XY, A{-}B{-}X \wedge A{-}C{-}Y \wedge X{-}T{-}Y$
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Versions of Pasch's axiom

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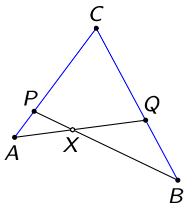
Axiom (Inner Pasch)

$$A—P—C \wedge B—Q—C \Rightarrow \exists X, P—X—B \wedge Q—X—A$$



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Axiom (Inner Pasch)

$$A{-}P{-}C \wedge B{-}Q{-}C \Rightarrow \exists X, P{-}X{-}B \wedge Q{-}X{-}A$$

Axiom (Outer Pasch)

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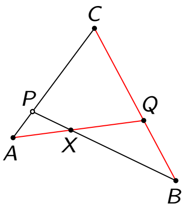

Gupta's independence model for Pasch

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- Points: $\mathbb{F}^2$ where $\mathbb{F}$ is a real closed field.

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Gupta's independence model for Pasch

- Points: $\mathbb{F}^2$ where $\mathbb{F}$ is a real closed field.
- $AB \equiv CD := (x_A - x_B)^2 + (y_A - y_B)^2 = (x_C - x_D)^2 + (y_C - y_D)^2$.
- A—B—$C := \exists k, 0 \leq k \leq 1 \wedge B - A = k(C - A)$ at the exception of the cases where $A = B$ and both A and C belong to the x-axis.

Szczerba's axiom system

Szczerba's axiom system

Identity for betweenness	
Inner transitivity for betweenness	
Outer transitivity for betweenness	
Transitivity for congruence	
Reflexivity for congruence	
Identity for congruence	
Segment Construction	
Pasch	$A—X—Q \wedge C—Q—B \Rightarrow \exists P, A—P—C \wedge B—X—P$
Five-Segment	
Lower 2-Dimensional	
Upper 2-Dimensional	
Euclid	$\neg(A—B—C \vee B—C—A \vee C—A—B) \Rightarrow$ $\exists C_C, AC_C \equiv BC_C \wedge AC_C \equiv CC_C$
Continuity	

Szczerba's axiom system

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Triangle circumscription principle

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Axiom (Triangle circumscription principle)

$$\neg(A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B) \Rightarrow$$
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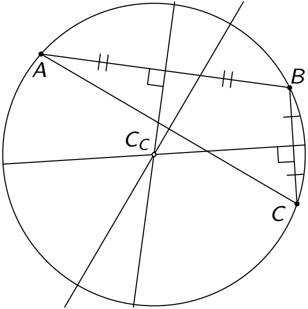

Outline

Parallel postulates are not *equivalent*

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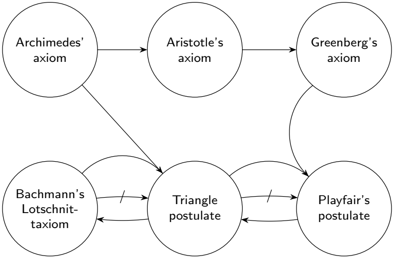

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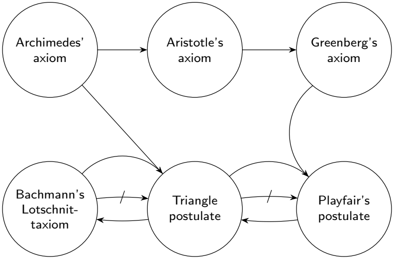

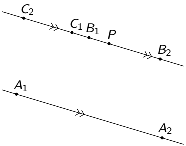

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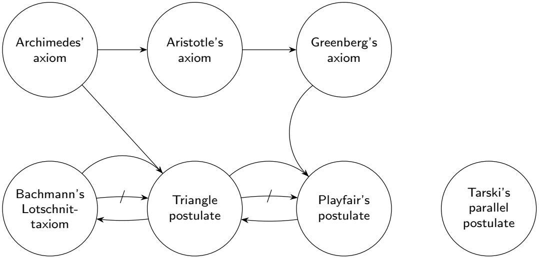

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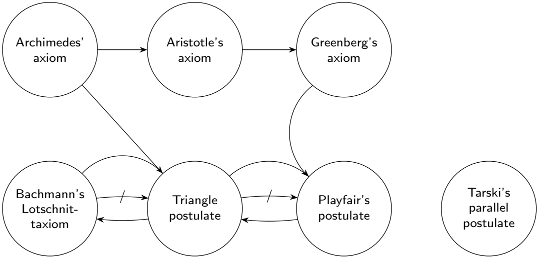

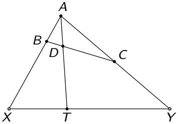

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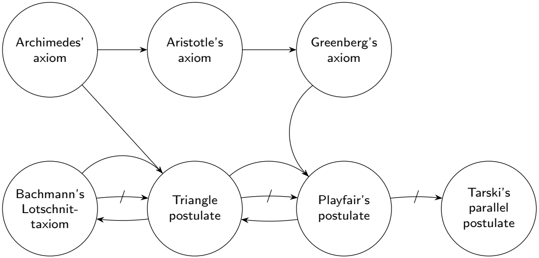

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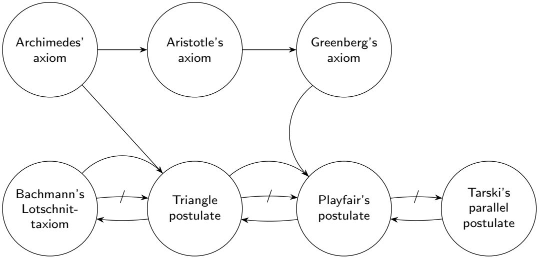

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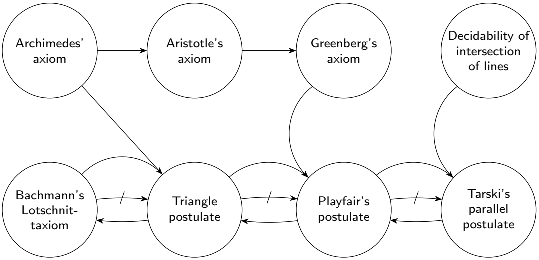

How to classify the postulates?

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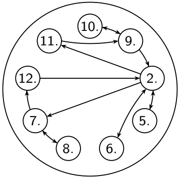

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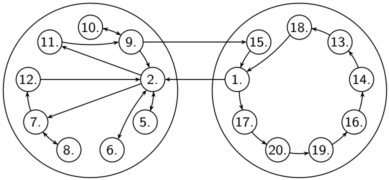

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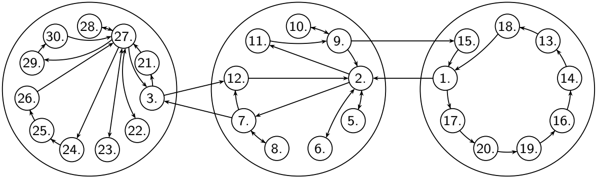

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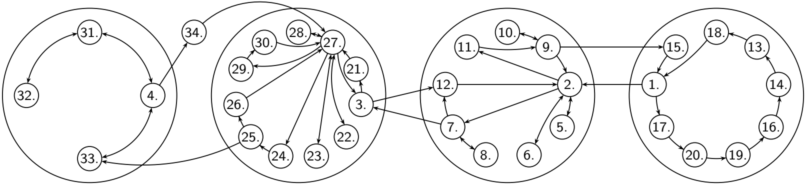

How to classify the postulates?

> *Pursuing the project faithfully will require that we take the extreme measure of shutting out the entreaties of our intuitions and imaginations - a forced separation of mental powers that will quite understandably be confusing and difficult to maintain [...].*

(Richard J. Trudeau)

The axioms

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Transitivity for betweenness	$A—B—D \wedge B—C—D \Rightarrow A—B—C$
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Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A—B—C \wedge A'—B'—C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
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Segment Construction	$\exists E, A$—B—$E \wedge BE \equiv CD$
Pasch	A—P—$C \wedge B$—Q—$C \wedge A \neq P \wedge P \neq C \wedge$ $B \neq Q \wedge Q \neq C \wedge \neg(A$—$B$—$C \vee B$—$C$—$A \vee C$—$A$—$B) \Rightarrow$ $\exists X, P$—X—$B \wedge Q$—X—A
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ A—B—$C \wedge A'$—B'—$C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A$—B—$C \wedge \neg B$—C—$A \wedge \neg C$—A—B
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ A—B—$C \vee B$—C—$A \vee C$—A—B
Proclus	
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A$—$X$—$Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X$—$B$—$Y \vee X = B \vee B = Y)$
Point equality decidability	$X = Y \vee X \neq Y$

A few definitions

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Collinearity

$$A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$$

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Coplanarity

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Strict parallelism

$$A \neq B \wedge C \neq D \wedge \text{Cp } A\,B\,C\,D \wedge \neg\exists X, \text{Col } A\,B\,X \wedge \text{Col } C\,D\,X$$

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Parallelism

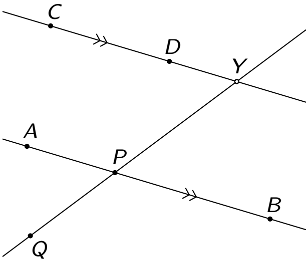
$$AB \parallel_s CD \vee (A \neq B \wedge C \neq D \wedge \text{Col } A\,C\,D \wedge \text{Col } B\,C\,D)$$

Proclus' axiom

Proclus' axiom

Axiom (Proclus' axiom)

$$AB \parallel CD \wedge \text{Col } A\,B\,P \wedge \neg \text{Col } A\,B\,Q \Rightarrow \exists Y, \text{Col } C\,D\,Y \wedge \text{Col } P\,Q\,Y$$



The axioms

The axioms

Symmetry for betweenness	$A\text{—}B\text{—}C \Rightarrow C\text{—}B\text{—}A$
Transitivity for betweenness	$A\text{—}B\text{—}D \wedge B\text{—}C\text{—}D \Rightarrow A\text{—}B\text{—}C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \wedge A \neq P \wedge P \neq C \wedge$ $B \neq Q \wedge Q \neq C \wedge \neg(A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B) \Rightarrow$ $\exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Proclus	
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X\text{—}B\text{—}Y \vee X = B \vee B = Y)$
Point equality decidability	$X = Y \vee X \neq Y$

The axioms

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- Tarski's system of geometry,
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The axioms

We have formalized that:

- Tarski's system of geometry,
- Gupta's axiom system,
- and this set of axioms are equivalent.

The axioms

Point equality decidability	$X = Y \lor X \neq Y$
Reflexivity for congruence	$AB \equiv BA$
Transitivity for congruence	$AB \equiv EF \land CD \equiv EF \Rightarrow AB \equiv CD$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \land BE \equiv CD$
Five-Segment	$AB \equiv A'B' \land BC \equiv B'C' \land AD \equiv A'D' \land BD \equiv B'D' \land$ $A\text{—}B\text{—}C \land A'\text{—}B'\text{—}C' \land A \neq B \Rightarrow CD \equiv C'D'$
Pasch	$A\text{—}P\text{—}C \land B\text{—}Q\text{—}C \land A \neq P \land P \neq C \land$ $B \neq Q \land Q \neq C \land \neg(A\text{—}B\text{—}C \lor B\text{—}C\text{—}A \lor C\text{—}A\text{—}B) \Rightarrow$ $\exists X, P\text{—}X\text{—}B \land Q\text{—}X\text{—}A$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \land \neg B\text{—}C\text{—}A \land \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \land$ $A \neq B \land B \neq C \land A \neq C \Rightarrow$ $A\text{—}B\text{—}C \lor B\text{—}C\text{—}A \lor C\text{—}A\text{—}B$
Proclus	
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \land \Upsilon Y \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \land \Upsilon Y \Rightarrow X\text{—}B\text{—}Y \lor X = B \lor B = Y)$
Symmetry for betweenness	$A\text{—}B\text{—}C \Rightarrow C\text{—}B\text{—}A$
Transitivity for betweenness	$A\text{—}B\text{—}D \land B\text{—}C\text{—}D \Rightarrow A\text{—}B\text{—}C$

The axioms

A0	$X = Y \vee X \neq Y$
A1	$AB \equiv BA$
A2'	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
A3	$AB \equiv CC \Rightarrow A = B$
A4	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
A5	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
A7'	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \wedge A \neq P \wedge P \neq C \wedge$ $B \neq Q \wedge Q \neq C \wedge \neg(A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B) \Rightarrow$ $\exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
A8	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
A9'	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
A10'	
A11'	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X\text{—}B\text{—}Y \vee X = B \vee B = Y)$
A14	$A\text{—}B\text{—}C \Rightarrow C\text{—}B\text{—}A$
A15	$A\text{—}B\text{—}D \wedge B\text{—}C\text{—}D \Rightarrow A\text{—}B\text{—}C$

Results

Results

	A0	A1	A2'	A3	A4	A5	A7'	A8	A9'	A10'	A11'	A14	A15
A0	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
A1	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
A2'	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
A3	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓
A4	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓
A5	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓
A7'	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓
A8	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓
A9'	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓
A10'	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓
A11'	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓
A14	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓
A15	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗

Results

	A0	A1	A2'	A3	A4	A5	A7'	A8	A9'	A10'	A11'	A14	A15
A0													
A1	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
A2'	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
A3	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓
A4	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓
A5	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓
A7'	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓
A8	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓
A9'	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓
A10'	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓
A11'	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓
A14	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓
A15	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗

Results

	A0	A1	A2'	A3	A4	A5	A7'	A8	A9'	A10'	A11'	A14	A15
A0													
A1	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
A2'	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
A3	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓
A4	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓
A5	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓
A7'	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓
A8	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓
A9'	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓
A10'	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓
A11'													
A14	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓
A15	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗

Results

	A0	A1	A2’	A3	A4	A5	A7’	A8	A9’	A10’	A11’	A14	A15
A0													
A1	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
A2’	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
A3	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓
A4	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓
A5	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓
A7’													
A8	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓
A9’	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓
A10’	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓
A11’													
A14	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓
A15	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗

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A0													
A1	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓		✓	✓
A2'	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
A3	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓
A4	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓
A5	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓		✓	✓
A7'													
A8	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓
A9'	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓		✓	✓
A10'	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗		✓	✓
A11'													
A14	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓		✗	✓
A15	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗

Results

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- An independent version of Tarski's system of geometry: formalization of 10 out of 13 counter-models in Coq.

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- We had to correct one of Gupta's models. We found a mistake in the communication between Tarski and Givant about *Tarski's System of Geometry*.

Thank you!