# Towards an independent version of Tarski's system of geometry

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ADG 2023 Belgrade, September 21, 2023

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Tarski's system of geometry Gupta's and Szczerba's contributions An independent version of Tarski's system of geometry? Conclusion



• Euclid's *Elements* 

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• Euclid's *Elements* led to what is likely the first independence proof.



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- Hilbert's Grundlagen der Geometrie



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- Tarski's System of Geometry



- Euclid's *Elements* led to what is likely the first independence proof.
- Hilbert's *Grundlagen der Geometrie* contains a chapter dedicated to independence properties.
- Tarski's System of Geometry and the problem of its independence was carefully studied by Gupta.

Tarski's system of geometry Gupta's and Szczerba's contributions An independent version of Tarski's system of geometry? Conclusion





- 2 Tarski's system of geometry
- Gupta's and Szczerba's contributions
- An independent version of Tarski's system of geometry?

## **5** Conclusion





- 2 Tarski's system of geometry
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5 Conclusion

Tarski's system of geometry

Gupta's and Szczerba's contributions An independent version of Tarski's system of geometry? Conclusior The axioms A model of the theory

The axioms A model of the theory

Tarski's system of geometry

• A single primitive type: point.



Alfred Tarski (1901 - 1983)

The axioms A model of the theory

- A single primitive type: point.
- Two primitive predicates:



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The axioms A model of the theory

- A single primitive type: point.
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  - **1** congruence  $AB \equiv CD$ ;
  - 2 betweenness A = B = C.
- 11 axioms.



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The axioms A model of the theory

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- 11 axioms.
- A parameter controls the dimension.



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- A single primitive type: point.
- Two primitive predicates:
  - congruence  $AB \equiv CD$ ;
  - 2 betweenness A = B = C.
- 11 axioms.
- A parameter controls the dimension.
- Good meta-theoritical properties.



Alfred Tarski (1901 - 1983)

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# Pasch's axiom

### Axiom (Pasch)

### $A - P - C \land B - Q - C \Rightarrow \exists X, P - X - B \land Q - X - A$



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Euclid's axiom

Axiom (Euclid)

$$A-D-T \land B-D-C \land A \neq D \Rightarrow$$
$$\exists XY, A-B-X \land A-C-Y \land X-T-Y$$

The axioms



Euclid's axiom

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Γ	he axioms		
	Identity for betweenness	А—В—	$-A \Rightarrow A = B$
	Transitivity for congruence	$AB \equiv$	$CD \land AB \equiv EF \Rightarrow CD \equiv EF$
	Reflexivity for congruence	$AB \equiv$	BA
	Identity for congruence	$AB \equiv$	$CC \Rightarrow A = B$
	Segment Construction	∃ <i>E</i> , <i>A</i> -	$-B-E \wedge BE \equiv CD$
	Pasch	A - P -	$-C \land B - Q - C \Rightarrow \exists X, P - X - B \land Q - X - A$
	Five-Segment	$AB \equiv$	$A'B' \wedge BC \equiv B'C' \wedge$
		$AD \equiv$	$A'D' \wedge BD \equiv B'D' \wedge$
		А—В—	$-C \wedge A' - B' - C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
	Lower 2-Dimensional	∃ABC	$, \neg A - B - C \land \neg B - C - A \land \neg C - A - B$
	Upper 2-Dimensional	$AP \equiv$	$AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \Rightarrow$
		А—В—	$-C \lor B - C - A \lor C - A - B$
	Euclid	A - D -	$-T \land B - D - C \land A \neq D \Rightarrow$
		$\exists XY, I$	$A - B - X \land A - C - Y \land X - T - Y$
	Continuity	∀ΞΥ,(	$(\exists A, (\forall XY, \exists X \land \Upsilon Y \Rightarrow A - X - Y)) \Rightarrow$
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Introduction <b>Tarski's system of geometry</b> Gupta's and Szczerba's contributions An independent version of Tarski's system of geometry? Conclusion		oduction cometry ributions cometry? onclusion	<b>The axioms</b> A model of the theory
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 $\begin{array}{l} \mathbf{y} \quad \forall = 1 \,, (\exists A, (\forall XY, \equiv X \land \top Y \Rightarrow A - X - Y)) \Rightarrow \\ \exists B, (\forall XY, \equiv X \land \Upsilon Y \Rightarrow X - B - Y) \end{array}$ 

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**Point equality decidability**  $X = Y \lor X \neq Y$ 

Tarski's system of geometry

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# Overview of the formalization

The axioms A model of the theory

## Overview of the formalization

W. Schwabhäuser W. Szmielew A. Tarski

#### Metamathematische Methoden in der Geometrie

Mit 167 Abbildungen

Teil I: Ein axiomatischer Aufbau der euklidischen Geometrie von W. Schwebhäuse, W. Sznietew und A. Tarski

Teil II: Metamathematische Betrachtungen von W. Schwabhäuser



Springer-Verlag Berlin Heidelberg New York Tokyo 1983

#### geocoq.github.io/GeoCoq/

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# A model of the theory
The axioms A model of the theory

## A model of the theory

• Points:  $\mathbb{F}^2$  where  $\mathbb{F}$  is a real closed field.

The axioms A model of the theory

# A model of the theory

 $\bullet$  Points:  $\mathbb{F}^2$  where  $\mathbb F$  is a real closed field.

• 
$$AB \equiv CD := (x_A - x_B)^2 + (y_A - y_B)^2 = (x_C - x_D)^2 + (y_C - y_D)^2$$
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The axioms A model of the theory

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•  $A - B - C := \exists k, 0 \le k \le 1 \land B - A = k(C - A).$ 

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.

• 
$$A - B - C := \exists k, 0 \le k \le 1 \land B - A = k(C - A).$$

• We mechanized the proof that all the axioms, excluding continuity, hold in *this* model.

Outline



- 2 Tarski's system of geometry
- 3 Gupta's and Szczerba's contributions
- 4 An independent version of Tarski's system of geometry?

#### 5 Conclusion

Gupta's contribution Szczerba's contribution

#### Gupta's axiom system

	Introd Tarski's system of geo <b>Gupta's and Szczerba's contrib</b> An independent version of Tarski's system of geon Conc	uction ometry utions netry? clusion	Gupta's contribution Szczerba's contribution
Gı	upta's axiom system		
	Identity for betweenness	A—B-	$-A \Rightarrow A = B$
	Transitivity for congruence	$AB \equiv$	$ECD \land AB \equiv EF \Rightarrow CD \equiv EF$
	Reflexivity for congruence	$AB \equiv$	BA
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	Segment Construction	$\exists E, A$	$-B-E \wedge BE \equiv CD$
	Pasch	A—P-	$-C \land B - Q - C \Rightarrow \exists X, P - X - B \land Q - X - A$
	Five-Segment	$AB \equiv$	$A'B' \wedge BC \equiv B'C' \wedge$
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		A—B-	$-C \land A' - B' - C' \land A \neq B \Rightarrow CD \equiv C'D'$
	Lower 2-Dimensional	∃ABC	$C, \neg A - B - C \land \neg B - C - A \land \neg C - A - B$
	Upper 2-Dimensional	$AP \equiv$	$AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \Rightarrow$
		A—B-	$-C \lor B - C - A \lor C - A - B$
	Euclid	A—D-	$-T \land B - D - C \land A \neq D \Rightarrow$
		$\exists XY,$	$A = B = X \land A = C = Y \land X = T = Y$
	Continuity	∀ΞΥ,	$(\exists A, (\forall XY, \exists X \land \Upsilon Y \Rightarrow A - X - Y)) \Rightarrow$

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Gι	ipta's axiom system		
	Transitivity for betweenness	A_B	$-D \land B - C - D \Rightarrow A - B - C$
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	Reflexivity for congruence	AB ≡	E BA
	Identity for congruence	AB ≡	$E CC \Rightarrow A = B$
	Segment Construction	$\exists E, A$	$-B-E \land BE \equiv CD$
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	Lower 2-Dimensional	∃AB¢	$C, \neg A - B - C \land \neg B - C - A \land \neg C - A - B$
	Upper 2-Dimensional	AP ≡	$AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \Rightarrow$

	$A - B - C \lor B - C - A \lor C - A - B$
Euclid	$A - D - T \land B - D - C \land A \neq D \Rightarrow$
	$\exists XY, A = B = X \land A = C = Y \land X = T = Y$
Continuity	$\forall \exists \Upsilon, (\exists A, (\forall XY, \exists X \land \Upsilon Y \Rightarrow A - X - Y)) \Rightarrow$
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	Lower 2-Dimensional	∃ABC	$C, \neg A - B - C \land \neg B - C - A \land \neg C - A - B$
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G	upta's axiom system		
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	Identity for congruence	$AB \equiv$	$CC \Rightarrow A = B$
	Segment Construction	∃ <i>E</i> , A	$-B-E \wedge BE \equiv CD$
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	Five-Segment	$AB \equiv$	$A'B' \wedge BC \equiv B'C' \wedge$
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	Lower 2-Dimensional	∃ABC	$C, \neg A - B - C \land \neg B - C - A \land \neg C - A - B$
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	Lower 2-Dimensional	∃ABO	$C, \neg A - B - C \land \neg B - C - A \land \neg C - A - B$
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	Transitivity for congruence	$AB \equiv EF \land CD \equiv EF \Rightarrow AB \equiv CD$	El	
	Reflexivity for congruence	$AB \equiv BA$		
	Identity for congruence	$AB \equiv CC \Rightarrow A = B$	В	
	Segment Construction	$\exists E, A - B - E \land BE \equiv CD$	$\equiv$	
	Pasch	$A - P - C \land B - Q - C \Rightarrow \exists X, P - X - B \land Q - X - A$	C =	Ā
	Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$	= 1	
		$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$	=	
		$A - B - C \land A' - B' - C' \land A \neq B \Rightarrow CD \equiv C'D'$	-C	
	Lower 2-Dimensional	$\exists ABC, \neg A - B - C \land \neg B - C - A \land \neg C - A - B$	\ -	
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		$A-B-C \lor B-C-A \lor C-A-B$	4 \	
	Euclid	$A - D - T \land B - D - C \land A \neq D \Rightarrow$	C.	
		$\exists XY, A = B = X \land A = C = Y \land X = T = Y$	_	
	Continuity	$\forall \exists \Upsilon, (\exists A, (\forall XY, \exists X \land \Upsilon Y \Rightarrow A - X - Y)) \Rightarrow$	ΞX	
		$\exists B, (\forall XY, \Xi X \land \Upsilon Y \Rightarrow X - B - Y)$	Ϋ́	

A	Intro Tarski's system of ge <b>Gupta's and Szczerba's contri</b> n independent version of Tarski's system of gec Cor	eometry ibutions Szczerba's contribution ometry? nclusion
Gu	pta's axiom system	
	Transitivity for betweenness	$A - B - D \land B - C - D \Rightarrow A - B - C$
	Transitivity for congruence	$AB \equiv EF \land CD \equiv EF \Rightarrow AB \equiv CD$
	Reflexivity for congruence	$AB \equiv BA$
	Identity for congruence	$AB \equiv CC \Rightarrow A = B$
	Segment Construction	$\exists E, A - B - E \land BE \equiv CD$
	Pasch	$A - P - C \land B - Q - C \Rightarrow \exists X, P - X - B \land Q - X - A$
	Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$
		$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$
		$A - B - C \land A' - B' - C' \land A \neq B \Rightarrow CD \equiv C'D'$
	Lower 2-Dimensional	$\exists ABC, \neg A - B - C \land \neg B - C - A \land \neg C - A - B$
	Upper 2-Dimensional	$AP \equiv AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \land$
		$A \neq B \land B \neq C \land A \neq C \Rightarrow$
		$A - B - C \lor B - C - A \lor C - A - B$
	Euclid	$A - D - T \land B - D - C \land A \neq D \Rightarrow$
		$\exists XY, A = B = X \land A = C = Y \land X = T = Y$
	Continuity	$\forall \exists \Upsilon, (\exists A, (\forall XY, \exists X \land \Upsilon Y \Rightarrow A - X - Y)) \Rightarrow$
		$\exists B, (\forall XY, \exists X \land \Upsilon Y \Rightarrow X - B - Y)$

	Introc Tarski's system of ge <b>Gupta's and Szczerba's contrib</b> An independent version of Tarski's system of geo Con	duction sometry Gupta's contribution ibutions Szczerba's contribution pmetry nclusion
Ĝ	upta's axiom system	
	Transitivity for betweenness	$A - B - D \land B - C - D \Rightarrow A - B - C$
	Transitivity for congruence	$AB \equiv EF \land CD \equiv EF \Rightarrow AB \equiv CD$
	Reflexivity for congruence	$AB \equiv BA$
	Identity for congruence	$AB \equiv CC \Rightarrow A = B$
	Segment Construction	$\exists E, A = B = CD$
	Pasch	$A - P - C \land B - Q - C \Rightarrow \exists X, P - X - B \land Q - X - A$
	Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$
		$AD \equiv A'D' \land BD \equiv B'D' \land$
		$A - B - C \land A' - B' - C' \land A \neq B \Rightarrow CD \equiv C'D'$
	Lower 2-Dimensional	$\exists ABC, \neg A - B - C \land \neg B - C - A \land \neg C - A - B$
	Upper 2-Dimensional	$AP \equiv AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \land$
		$A \neq B \land B \neq C \land A \neq C \Rightarrow$
		$A - B - C \lor B - C - A \lor C - A - B$
	Euclid	$A - D - T \land B - D - C \land A \neq D \Rightarrow$
		$\exists XY, A = B = X \land A = C = Y \land X = T = Y$
	Continuity	$\forall = \mathrm{T}, (\exists A, (\forall XY, =X \land \mathrm{T}Y \Rightarrow A - X - Y)) \Rightarrow$
		$\exists B, (\forall XY, \exists X \land \top Y \Rightarrow$
		$X - B - Y \lor X = B \lor B = Y$

	Introd Tarski's system of ged <b>Gupta's and Szczerba's contrib</b> An independent version of Tarski's system of geor Con	duction ometry outions metry? clusion	Gupta's contribution Szczerba's contribution
G	upta's axiom system		
	Transitivity for betweenness	A—B-	$-D \land B-C-D \Rightarrow A-B-C$
	Transitivity for congruence	$AB \equiv$	$EF \land CD \equiv EF \Rightarrow AB \equiv CD$
	Reflexivity for congruence	$AB \equiv$	BA
	Identity for congruence	$AB \equiv$	$CC \Rightarrow A = B$
	Segment Construction	$\exists E, A$	$-B-E \land BE \equiv CD$
	Pasch	A - P -	$-C \land B - Q - C \Rightarrow \exists X, P - X - B \land Q - X - A$
	Five-Segment	$AB \equiv$	$A'B' \wedge BC \equiv B'C' \wedge$
		$AD \equiv$	$A'D' \wedge BD \equiv B'D' \wedge$
		A - B -	$-C \land A' - B' - C' \land A \neq B \Rightarrow CD \equiv C'D'$
	Lower 2-Dimensional	∃ABC	$C, \neg A - B - C \land \neg B - C - A \land \neg C - A - B$
	Upper 2-Dimensional	$AP \equiv$	$AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \land$
		$A \neq E$	$B \land B \neq C \land A \neq C \Rightarrow$
		A—B-	$-C \lor B - C - A \lor C - A - B$
	Euclid	A—D-	$-T \land B - D - C \land A \neq D \Rightarrow$
		$\exists XY,$	$A - B - X \land A - C - Y \land X - T - Y$
	Continuity	∀∃'n,	$(\exists A, (\forall XY, \exists X \land \exists Y \Rightarrow A - X - Y)) \Rightarrow$
		∃ <i>B</i> ,(∖	$\forall XY, = X \land TY \Rightarrow$
			$X - B - Y \lor X = B \lor B = Y$

	Introc Tarski's system of ge <b>Gupta's and Szczerba's contril</b> An independent version of Tarski's system of geo Con	duction ometry outions metry? clusion	Gupta's contribution Szczerba's contribution		
Gupta's axiom system					
	Transitivity for betweenness	A—B-	$-D \land B-C-D \Rightarrow A-B-C$		
	Transitivity for congruence	$AB \equiv$	$E EF \land CD \equiv EF \Rightarrow AB \equiv CD$		
	Reflexivity for congruence	$AB \equiv$	E BA		
	Identity for congruence	$AB \equiv$	$= CC \Rightarrow A = B$		
	Segment Construction	∃ <i>E</i> , A	$-B-E \wedge BE \equiv CD$		
	Pasch	A - X	$-Q \land C - Q - B \Rightarrow \exists P, A - P - C \land B - X - P$		
	Five-Segment	$AB \equiv$	$A'B' \wedge BC \equiv B'C' \wedge$		
		$AD \equiv$	$\equiv A'D' \wedge BD \equiv B'D' \wedge$		
		A—B-	$-C \land A' - B' - C' \land A \neq B \Rightarrow CD \equiv C'D'$		
	Lower 2-Dimensional	∃ABC	$C, \neg A - B - C \land \neg B - C - A \land \neg C - A - B$		
	Upper 2-Dimensional	$AP \equiv$	$AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \land$		
		$A \neq I$	$B \land B \neq C \land A \neq C \Rightarrow$		
		A—B-	$-C \lor B - C - A \lor C - A - B$		
	Euclid	A - D	$-T \land B - D - C \land A \neq D \Rightarrow$		
		$\exists XY,$	$A - B - X \land A - C - Y \land X - T - Y$		
	Continuity	∀ΞΥ,	$(\exists A, (\forall XY, \exists X \land \Upsilon Y \Rightarrow A - X - Y)) \Rightarrow$		
		∃ <i>B</i> , (`	$\forall XY, \exists X \land \Upsilon Y \Rightarrow$		
			$X - B - Y \lor X = B \lor B = Y)$		

Gupta's contribution Szczerba's contribution

## Versions of Pasch's axiom

Gupta's contribution Szczerba's contribution

## Versions of Pasch's axiom

#### Axiom (Inner Pasch)

$$A - P - C \land B - Q - C \Rightarrow \exists X, P - X - B \land Q - X - A$$



Gupta's contribution Szczerba's contribution

## Versions of Pasch's axiom

#### Axiom (Inner Pasch)

$$A - P - C \land B - Q - C \Rightarrow \exists X, P - X - B \land Q - X - A$$

#### Axiom (Outer Pasch)

$$A = X = Q \land C = Q = B \Rightarrow \exists P, A = P = C \land B = X = P$$



Gupta's contribution Szczerba's contribution

## Gupta's independence model for Pasch

Gupta's contribution Szczerba's contribution

## Gupta's independence model for Pasch

• Points:  $\mathbb{F}^2$  where  $\mathbb{F}$  is a real closed field.

Gupta's contribution Szczerba's contribution

#### Gupta's independence model for Pasch

• Points:  $\mathbb{F}^2$  where  $\mathbb{F}$  is a real closed field.

• 
$$AB \equiv CD := (x_A - x_B)^2 + (y_A - y_B)^2 = (x_C - x_D)^2 + (y_C - y_D)^2$$
.

Gupta's contribution Szczerba's contribution

#### Gupta's independence model for Pasch

• Points:  $\mathbb{F}^2$  where  $\mathbb{F}$  is a real closed field.

• 
$$AB \equiv CD := (x_A - x_B)^2 + (y_A - y_B)^2 = (x_C - x_D)^2 + (y_C - y_D)^2$$
.

A-B-C := ∃k, 0 ≤ k ≤ 1 ∧ B − A = k(C − A) at the exception of the cases where A = B and both A and C belong to the x-axis.

Gupta's contribution Szczerba's contribution

## Szczerba's axiom system

Gupta's contribution Szczerba's contribution

#### Szczerba's axiom system

Identity for betweenness	
Inner transitivity for betweenness	
Outer transitivity for betweenness	
Transitivity for congruence	
Reflexivity for congruence	
Identity for congruence	
Segment Construction	
Pasch	$A - X - Q \land C - Q - B \Rightarrow \exists P, A - P - C \land B - X - P$
Five-Segment	
Lower 2-Dimensional	
Upper 2-Dimensional	
Euclid	$\neg(A\_B\_C \lor B\_C\_A \lor C\_A\_B) \Rightarrow$
	$\exists C_C, AC_C \equiv BC_C \land AC_C \equiv CC_C$
Continuity	

Gupta's contribution Szczerba's contribution

#### Szczerba's axiom system

Identity for betweenness	
Inner transitivity for betweenness	
Outer transitivity for betweenness	
Transitivity for congruence	
Reflexivity for congruence	
Identity for congruence	
Segment Construction	
Pasch	$A - X - Q \land C - Q - B \Rightarrow \exists P, A - P - C \land B - X - P$
Five-Segment	
Lower 2-Dimensional	
Upper 2-Dimensional	
Euclid	$\neg (A - B - C \lor B - C - A \lor C - A - B) \Rightarrow$
	$\exists C_C, AC_C \equiv BC_C \land AC_C \equiv CC_C$
Continuity	

Gupta's contribution Szczerba's contribution

# Triangle circumscription principle

Gupta's contribution Szczerba's contribution

# Triangle circumscription principle

#### Axiom (Triangle circumscription principle)

$$\neg (A - B - C \lor B - C - A \lor C - A - B) \Rightarrow \exists C_C, A C_C \equiv B C_C \land A C_C \equiv C C_C$$

Gupta's contribution Szczerba's contribution

# Triangle circumscription principle

#### Axiom (Triangle circumscription principle)

$$\neg (A - B - C \lor B - C - A \lor C - A - B) \Rightarrow \exists C_C, AC_C \equiv BC_C \land AC_C \equiv CC_C$$





Parallel postulates are not *equivalent* How to classify the postulates? The axioms

#### 1 Introduction

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Parallel postulates are not *equivalent* How to classify the postulates? The axioms



Parallel postulates are not *equivalent* How to classify the postulates? The axioms

#### Parallel postulates are not equivalent





Parallel postulates are not *equivalent* How to classify the postulates? The axioms



Parallel postulates are not *equivalent* How to classify the postulates? The axioms



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## Parallel postulates are not equivalent



Parallel postulates are not *equivalent* How to classify the postulates? The axioms

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Parallel postulates are not *equivalent* How to classify the postulates? The axioms

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Parallel postulates are not *equivalent* How to classify the postulates? The axioms

# How to classify the postulates?

Pursuing the project faithfully will require that we take the extreme measure of shutting out the entreaties of our intuitions and imaginations - a forced separation of mental powers that will quite understandably be confusing and difficult to maintain [...].

(Richard J. Trudeau)



Parallel postulates are not *equivalent* How to classify the postulates? The axioms Introduction Tarski's system of geometry Gupta's and Szczerba's contributions An independent version of Tarski's system of geometry? Conclusion Conclusion Parallel postulates are not equivalent How to classify the postulates? The axioms

## The axioms

Transitivity for betweenness	$A - B - D \land B - C - D \Rightarrow A - B - C$
Transitivity for congruence	$AB \equiv EF \land CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A - B - E \land BE \equiv CD$
Pasch	$A - X - Q \land C - Q - B \Rightarrow \exists P, A - P - C \land B - X - P$

Five-Segment	$AB \equiv A'B' \land BC \equiv B'C' \land AD \equiv A'D' \land BD \equiv B'D' \land$
	$A - B - C \land A' - B' - C' \land A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A = B = C \land \neg B = C = A \land \neg C = A = B$
Upper 2-Dimensional	$AP \equiv AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \land$
	$A \neq B \land B \neq C \land A \neq C \Rightarrow$
	$A - B - C \lor B - C - A \lor C - A - B$
Euclid	$A - D - T \land B - D - C \land A \neq D \Rightarrow$
	$\exists XY, A = B = X \land A = C = Y \land X = T = Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \land \Upsilon Y \Rightarrow A - X - Y)) \Rightarrow$
	$\exists B, (\forall XY, \Xi X \land \Upsilon Y \Rightarrow X - B - Y \lor X = B \lor B = Y)$

Introduction Tarski's system of geometr Gupta's and Szczerba's contribution: An independent version of Tarski's system of geometry Conclusion	Parallel postulates are not <i>equivalent</i> s How to classify the postulates? <b>? The axioms</b>
The axioms	
Symmetry for betweenness A-B-	$C \Rightarrow C - B - A$
Transitivity for betweenness A-B-	$D \wedge B - C - D \Rightarrow A - B - C$
Transitivity for congruence $AB \equiv$	$EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence $AB \equiv$	BA
Identity for congruence $AB \equiv$	$CC \Rightarrow A = B$
Segment Construction $\exists E, A_{\neg}$	$-B-E \wedge BE \equiv CD$
Pasch A—X—	$-Q \land C - Q - B \Rightarrow \exists P, A - P - C \land B - X - P$

Five-Segment	$AB \equiv A'B' \land BC \equiv B'C' \land AD \equiv A'D' \land BD \equiv B'D' \land$
	$A - B - C \land A' - B' - C' \land A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A = B = C \land \neg B = C = A \land \neg C = A = B$
Upper 2-Dimensional	$AP \equiv AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \land$
	$A \neq B \land B \neq C \land A \neq C \Rightarrow$
	$A - B - C \lor B - C - A \lor C - A - B$
Euclid	$A - D - T \land B - D - C \land A \neq D \Rightarrow$
	$\exists XY, A = B = X \land A = C = Y \land X = T = Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \land \Upsilon Y \Rightarrow A - X - Y)) \Rightarrow$
	$\exists B, (\forall XY, \Xi X \land \Upsilon Y \Rightarrow X - B - Y \lor X = B \lor B = Y)$

Introduction Tarski's system of geometry Gupta's and Szczerba's contributions An independent version of Tarski's system of geometry? Conclusion	Parallel postulates are not <i>equivalent</i> How to classify the postulates? <b>The axioms</b>
The axioms	
Symmetry for betweenness A–B–C	$\Rightarrow C - B - A$
Transitivity for betweenness A—B—D	$\wedge B - C - D \Rightarrow A - B - C$
Transitivity for congruence $AB \equiv EF$	$F \land CD \equiv EF \Rightarrow AB \equiv CD$

Reflexivity for congruence  $AB \equiv BA$ 

Identity for congruence  $AB \equiv CC \Rightarrow A = B$ 

Segment Construction  $\exists E, A - B - E \land BE \equiv CD$ 

Pasch	$A = X = Q \land C = Q = B \Rightarrow \exists P, A = P = C \land B = X = P$	

$AB \equiv A'B' \land BC \equiv B'C' \land AD \equiv A'D' \land BD \equiv B'D' \land$
$A - B - C \land A' - B' - C' \land A \neq B \Rightarrow CD \equiv C'D'$
$\exists ABC, \neg A - B - C \land \neg B - C - A \land \neg C - A - B$
$AP \equiv AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \land$
$A \neq B \land B \neq C \land A \neq C \Rightarrow$
$A - B - C \lor B - C - A \lor C - A - B$
$A - D - T \land B - D - C \land A \neq D \Rightarrow$
$\exists XY, A = B = X \land A = C = Y \land X = T = Y$
$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \land \Upsilon Y \Rightarrow A - X - Y)) \Rightarrow$
$\exists B, (\forall XY, \Xi X \land \Upsilon Y \Rightarrow X - B - Y \lor X = B \lor B = Y)$

Introduction Tarski's system of geometry Gupta's and Szczerba's contributions An independent version of Tarski's system of geometry? Conclusion	Parallel postulates are not <i>equivalent</i> How to classify the postulates? <b>The axioms</b>
The axioms	
Symmetry for betweenness $A - B - C =$	$\Rightarrow C - B - A$
Transitivity for betweenness A—B—D	$\wedge B - C - D \Rightarrow A - B - C$

Symmetry for betweenness	$A - B - C \Rightarrow C - B - A$
Transitivity for betweenness	$A - B - D \land B - C - D \Rightarrow A - B - C$
Transitivity for congruence	$AB \equiv EF \land CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
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Segment Construction	$\exists E, A - B - E \land BE \equiv CD$
Pasch	$A - X - Q \land C - Q - B \Rightarrow \exists P, A - P - C \land B - X - P$

Five-Segment	$AB \equiv A'B' \land BC \equiv B'C' \land AD \equiv A'D' \land BD \equiv B'D' \land$
	$A - B - C \land A' - B' - C' \land A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A = B = C \land \neg B = C = A \land \neg C = A = B$
Upper 2-Dimensional	$AP \equiv AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \land$
	$A \neq B \land B \neq C \land A \neq C \Rightarrow$
	$A - B - C \lor B - C - A \lor C - A - B$
Euclid	$A - D - T \land B - D - C \land A \neq D \Rightarrow$
	$\exists XY, A = B = X \land A = C = Y \land X = T = Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \land \Upsilon Y \Rightarrow A - X - Y)) \Rightarrow$
	$\exists B, (\forall XY, \Xi X \land \Upsilon Y \Rightarrow X - B - Y \lor X = B \lor B = Y)$
Point equality decidability	$X = Y \lor X \neq Y$

Ar	Introduction Tarski's system of geometry Gupta's and Szczerba's contributions n independent version of Tarski's system of geometry? Conclusion	Parallel postulates are not <i>equivalent</i> How to classify the postulates? <b>The axioms</b>
Th	e axioms	
	Symmetry for betweenness A–B–C	$\Rightarrow C - B - A$
_	Transitivity for betweenness A–B–D	$\wedge B - C - D \Rightarrow A - B - C$

Symmetry for betweenness	$A - B - C \Rightarrow C - B - A$
Transitivity for betweenness	$A - B - D \land B - C - D \Rightarrow A - B - C$
Transitivity for congruence	$AB \equiv EF \land CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A - B - E \land BE \equiv CD$
Pasch	$A - X - Q \land C - Q - B \Rightarrow \exists P, A - P - C \land B - X - P$

Five-Segment	$AB \equiv A'B' \land BC \equiv B'C' \land AD \equiv A'D' \land BD \equiv B'D' \land$
	$A - B - C \land A' - B' - C' \land A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A = B = C \land \neg B = C = A \land \neg C = A = B$
Upper 2-Dimensional	$AP \equiv AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \land$
	$A \neq B \land B \neq C \land A \neq C \Rightarrow$
	$A - B - C \lor B - C - A \lor C - A - B$
Euclid	$A - D - T \land B - D - C \land A \neq D \Rightarrow$
	$\exists XY, A = B = X \land A = C = Y \land X = T = Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \land \Upsilon Y \Rightarrow A - X - Y)) \Rightarrow$
	$\exists B, (\forall XY, \Xi X \land \Upsilon Y \Rightarrow X - B - Y \lor X = B \lor B = Y)$
Point equality decidability	$X = Y \lor X \neq Y$

l Tarski's system Gupta's and Szczerba's co An independent version of Tarski's system of	Introduction of geometry ontributions <b>f geometry?</b> Conclusion	Parallel postulates are not <i>equivalent</i> How to classify the postulates? <b>The axioms</b>
The axioms		
Symmetry for betweenness	А—В—С	$\Rightarrow C - B - A$
Transitivity for betweenness	A—B—D	$\wedge B - C - D \Rightarrow A - B - C$
Transitivity for congruence	$AB \equiv EF$	$F \land CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$	4
Identity for congruence	$AB \equiv CC$	$C \Rightarrow A = B$
Segment Construction	∃ <i>E</i> , <i>A</i> — <i>B</i>	$-E \land BE \equiv CD$
Pasch	A_P_C	$\land B - Q - C \land A \neq P \land P \neq C \land$
	$R \neq 0$	$O \neq C \land \neg (A \land B \land C \lor B \land A \lor C \land B) \rightarrow$

	$D \neq Q \land Q \neq C \land (A = D = C \lor D = C = A \lor C = A = D) \Rightarrow$
	$\exists X, P - X - B \land Q - X - A$
Five-Segment	$AB \equiv A'B' \land BC \equiv B'C' \land AD \equiv A'D' \land BD \equiv B'D' \land$
	$A - B - C \land A' - B' - C' \land A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A = B = C \land \neg B = C = A \land \neg C = A = B$
Upper 2-Dimensional	$AP \equiv AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \land$
	$A \neq B \land B \neq C \land A \neq C \Rightarrow$
	$A - B - C \lor B - C - A \lor C - A - B$
Euclid	$A - D - T \land B - D - C \land A \neq D \Rightarrow$
	$\exists XY, A = B = X \land A = C = Y \land X = T = Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \land \Upsilon Y \Rightarrow A - X - Y)) \Rightarrow$
	$\exists B, (\forall XY, \Xi X \land \Upsilon Y \Rightarrow X - B - Y \lor X = B \lor B = Y)$
Point equality decidability	$X = Y \lor X \neq Y$

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Th	e axioms			
	Symmetry for betweenness	А—В—С	$\Rightarrow C - B - A$	
-	Transitivity for betweenness	A—B—D	$\wedge B - C - D \Rightarrow A - B - C$	
	Transitivity for congruence	$AB \equiv EF$	$F \land CD \equiv EF \Rightarrow AB \equiv CD$	
-	Reflexivity for congruence	$AB \equiv BA$	1	
-	Identity for congruence	$AB \equiv CC$	$C \Rightarrow A = B$	
	Segment Construction	∃ <i>E</i> , <i>A</i> — <i>B</i>	$-E \land BE \equiv CD$	

Segment Construction	$\exists E, A = D = E \land D E \equiv C D$
Pasch	$A - P - C \land B - Q - C \land A \neq P \land P \neq C \land$
	$B \neq Q \land Q \neq C \land \neg (A - B - C \lor B - C - A \lor C - A - B) \Rightarrow$
	$\exists X, P - X - B \land Q - X - A$
Five-Segment	$AB \equiv A'B' \land BC \equiv B'C' \land AD \equiv A'D' \land BD \equiv B'D' \land$
	$A - B - C \land A' - B' - C' \land A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A = B = C \land \neg B = C = A \land \neg C = A = B$
Upper 2-Dimensional	$AP \equiv AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \land$
	$A \neq B \land B \neq C \land A \neq C \Rightarrow$
	$A - B - C \lor B - C - A \lor C - A - B$
Euclid	$A - D - T \land B - D - C \land A \neq D \Rightarrow$
	$\exists XY, A = B = X \land A = C = Y \land X = T = Y$
Continuity	$\forall \exists \Upsilon, (\exists A, (\forall XY, \exists X \land \Upsilon Y \Rightarrow A - X - Y)) \Rightarrow$
	$\exists B, (\forall XY, \Xi X \land \Upsilon Y \Rightarrow X - B - Y \lor X = B \lor B = Y)$
Point equality decidability	$X = Y \lor X \neq Y$

Tarski's system Gupta's and Szczerba's o An independent version of Tarski's system o	Introduction of geometry contributions of geometry? Conclusion	Parallel postulates are not <i>equivalent</i> How to classify the postulates? <b>The axioms</b>
The axioms		
Symmetry for betweenness	А—В—С	$\Rightarrow$ C-B-A
Transitivity for betweenness	A—B—D	$\wedge B - C - D \Rightarrow A - B - C$
Transitivity for congruence	$AB \equiv EF$	$F \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$	4
Identity for congruence	$AB \equiv CC$	$C \Rightarrow A = B$
Segment Construction	∃ <i>E</i> , <i>A</i> — <i>B</i>	$E - E \wedge BE \equiv CD$
Pasch	A P C	$\land$ B-Q-C $\land$ A $\neq$ P $\land$ P $\neq$ C $\land$
	$B  eq Q \land$	$Q \neq C \land \neg (A - B - C \lor B - C - A \lor C - A - B) \Rightarrow$
	$\exists X, P - X$	$X = B \land Q = X = A$
Five-Segment	$AB \equiv A'$	$B' \land BC \equiv B'C' \land AD \equiv A'D' \land BD \equiv B'D' \land$
	A - B - C	$\wedge A' - B' - C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg$	$A - B - C \land \neg B - C - A \land \neg C - A - B$
Upper 2-Dimensional	$AP \equiv AG$	$Q \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \land$
	$A \neq B \land$	$B \neq C \land A \neq C \Rightarrow$
	A—B—C	$\lor B - C - A \lor C - A - B$

#### Proclus

Continuity	$ \forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \land \Upsilon Y \Rightarrow A - X - Y)) \Rightarrow \\ \exists B, (\forall XY, \Xi X \land \Upsilon Y \Rightarrow X - B - Y \lor X = B \lor B = Y) $
Point equality decidability	$X = Y \lor X \neq Y$

Parallel postulates are not *equivalent* How to classify the postulates? The axioms

# A few definitions

Parallel postulates are not *equivalent* How to classify the postulates? The axioms

## A few definitions

### Collinearity

## $A - B - C \lor B - C - A \lor C - A - B$

Parallel postulates are not *equivalent* How to classify the postulates? The axioms

## A few definitions

#### Collinearity

## $A - B - C \lor B - C - A \lor C - A - B$

#### Coplanarity

# $\exists X, (\mathsf{Col}\; A \, B \, X \land \mathsf{Col}\; C \, D \, X) \lor (\mathsf{Col}\; A \, C \, X \land \mathsf{Col}\; B \, D \, X) \lor \\ (\mathsf{Col}\; A \, D \, X \land \mathsf{Col}\; B \, C \, X)$

Parallel postulates are not *equivalent* How to classify the postulates? The axioms

# A few definitions

#### Collinearity

## $A - B - C \lor B - C - A \lor C - A - B$

#### Coplanarity

 $\exists X, (\mathsf{Col}\; A \, B \, X \land \mathsf{Col}\; C \, D \, X) \lor (\mathsf{Col}\; A \, C \, X \land \mathsf{Col}\; B \, D \, X) \lor \\ (\mathsf{Col}\; A \, D \, X \land \mathsf{Col}\; B \, C \, X)$ 

#### Strict parallelism

 $A \neq B \land C \neq D \land \mathsf{Cp} \ A B \ C \ D \land \neg \exists X, \mathsf{Col} \ A B \ X \land \mathsf{Col} \ C \ D \ X$ 

Parallel postulates are not *equivalent* How to classify the postulates? The axioms

# A few definitions

#### Collinearity

## $A - B - C \lor B - C - A \lor C - A - B$

#### Coplanarity

 $\exists X, (Col \ A \ B \ X \land Col \ C \ D \ X) \lor (Col \ A \ C \ X \land Col \ B \ D \ X) \lor (Col \ A \ C \ X \land Col \ B \ D \ X) \lor$ 

#### Strict parallelism

 $A \neq B \land C \neq D \land \mathsf{Cp} \ A B \ C \ D \land \neg \exists X, \mathsf{Col} \ A B \ X \land \mathsf{Col} \ C \ D \ X$ 

#### Parallelism

 $AB \parallel_{s} CD \lor (A \neq B \land C \neq D \land Col \ A \ C \ D \land Col \ B \ C \ D)$ 

Parallel postulates are not *equivalent* How to classify the postulates? The axioms

# Proclus' axiom

Parallel postulates are not *equivalent* How to classify the postulates? The axioms

## Proclus' axiom

#### Axiom (Proclus' axiom)

 $\begin{array}{l} AB \parallel CD \land \mathsf{Col} \ A B \ P \land \neg \mathsf{Col} \ A B \ Q \Rightarrow \\ \exists Y, \mathsf{Col} \ C \ D \ Y \land \mathsf{Col} \ P \ Q \ Y \end{array}$ 



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Parallel postulates are not *equivalent* How to classify the postulates? The axioms

## The axioms

Symmetry for betweenness	$A - B - C \Rightarrow C - B - A$
Transitivity for betweenness	$A - B - D \land B - C - D \Rightarrow A - B - C$
Transitivity for congruence	$AB \equiv EF \land CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A - B - E \land BE \equiv CD$
Pasch	$A - P - C \land B - Q - C \land A \neq P \land P \neq C \land$
	$B \neq Q \land Q \neq C \land \neg (A - B - C \lor B - C - A \lor C - A - B) \Rightarrow$
	$\exists X, P = X = B \land Q = X = A$
Five-Segment	$AB \equiv A'B' \land BC \equiv B'C' \land AD \equiv A'D' \land BD \equiv B'D' \land$
	$A - B - C \land A' - B' - C' \land A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A = B = C \land \neg B = C = A \land \neg C = A = B$
Upper 2-Dimensional	$AP \equiv AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \land$
	$A \neq B \land B \neq C \land A \neq C \Rightarrow$
	$A - B - C \lor B - C - A \lor C - A - B$
Proclus	
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \land \Upsilon Y \Rightarrow A = X = Y)) \Rightarrow$
	$\exists B, (\forall XY, \Xi X \land \Upsilon Y \Rightarrow X - B - Y \lor X = B \lor B = Y)$
Point equality decidability	$X = Y \lor X \neq Y$



Parallel postulates are not *equivalent* How to classify the postulates? The axioms

We have formalized that:





## We have formalized that:

• Tarski's system of geometry,

Parallel postulates are not *equivalent* How to classify the postulates? The axioms



We have formalized that:

- Tarski's system of geometry,
- Gupta's axiom system,

Parallel postulates are not *equivalent* How to classify the postulates? The axioms



We have formalized that:

- Tarski's system of geometry,
- Gupta's axiom system,
- and this set of axioms are equivalent.

lr Tarski's system o Gupta's and Szczerba's co An independent version of Tarski's system of	ntroduction f geometry Parallel postulates are not <i>equivalent</i> ntributions How to classify the postulates? geometry? The axioms Conclusion
The axioms	
Point equality decidability	$X = Y \lor X \neq Y$
Reflexivity for congruence	$AB \equiv BA$
Transitivity for congruence	$AB \equiv EF \land CD \equiv EF \Rightarrow AB \equiv CD$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A - B - E \land BE \equiv CD$
Five-Segment	$AB \equiv A'B' \land BC \equiv B'C' \land AD \equiv A'D' \land BD \equiv B'D' \land$
	$A - B - C \land A' - B' - C' \land A \neq B \Rightarrow CD \equiv C'D'$
Pasch	$A - P - C \land B - Q - C \land A \neq P \land P \neq C \land$
	$B \neq Q \land Q \neq C \land \neg (A - B - C \lor B - C - A \lor C - A - B) \Rightarrow$
	$\exists X, P = X = B \land Q = X = A$
Lower 2-Dimensional	$\exists ABC, \neg A = B = C \land \neg B = C = A \land \neg C = A = B$
Upper 2-Dimensional	$AP \equiv AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \land$
	$A \neq B \land B \neq C \land A \neq C \Rightarrow$
	$A - B - C \lor B - C - A \lor C - A - B$
Proclus	
Continuity	$\forall \exists T, (\exists A, (\forall XY, \exists X \land TY \Rightarrow A - X - Y)) \Rightarrow$
	$\exists B, (\forall XY, \exists X \land \top Y \Rightarrow X - B - Y \lor X = B \lor B = Y)$
Symmetry for betweenness	$A - B - C \Rightarrow C - B - A$
Transitivity for betweenness	$A - B - D \land B - C - D \Rightarrow A - B - C$

l Tarski's system o Gupta's and Szczerba's co An independent version of Tarski's system of	ntroduction of geometry ontributions f geometry? Conclusion	Parallel postulates are not <i>equivalent</i> How to classify the postulates? <b>The axioms</b>
The axioms		
A0	$X = Y \lor$	$X \neq Y$
A1	$AB \equiv BA$	
A2'	$AB \equiv EF$	$\wedge CD \equiv EF \Rightarrow AB \equiv CD$
A3	$AB \equiv CC$	$T \Rightarrow A = B$
A4	∃ <i>E</i> , <i>A</i> — <i>B</i> -	$-E \wedge BE \equiv CD$
A5	$AB \equiv A'I$	$B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \wedge$
	A - B - C	$\land A' - B' - C' \land A \neq B \Rightarrow CD \equiv C'D'$
A7'	A-P-C	$\land B - Q - C \land A \neq P \land P \neq C \land$
	$B  eq Q \land$	$Q \neq C \land \neg (A - B - C \lor B - C - A \lor C - A - B) \Rightarrow$
	$\exists X, P = X$	$-B \land Q - X - A$
A8	$\exists ABC, \neg A$	$A = B = C \land \neg B = C = A \land \neg C = A = B$
A9'	$AP \equiv AQ$	$P \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \land$
	$A \neq B \land A$	$B \neq C \land A \neq C \Rightarrow$
	A = B = C	$\lor B = C = A \lor C = A = B$
A10'		
A11'	∀∃Ƴ,(∃A	$(\forall XY, \exists X \land \Upsilon Y \Rightarrow A - X - Y)) \Rightarrow$
	∃ <i>B</i> ,(∀ <i>XY</i>	$Y, \exists X \land \Upsilon Y \Rightarrow X - B - Y \lor X = B \lor B = Y)$
A14	A - B - C	$\Rightarrow C - B - A$
A15	A = B = D	$\wedge B - C - D \Rightarrow A - B - C$



## Results

	A0	A1	A2'	A3	A4	A5	A7'	A8	A9'	A10'	A11'	A14	A15
A0	×	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>
A1	-	×	<ul> <li>Image: A second s</li></ul>	-	-	-	-	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	-
A2'	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>
A3	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>
A4	-	-	<ul> <li>Image: A second s</li></ul>	-	×	-	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	-
A5	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	-	- <u>-</u>
A7'	-	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	-	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>
A8	-	-	<ul> <li>Image: A second s</li></ul>	-	-	-	-	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>
A9'	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	-	- <u>-</u>
A10'	-	-	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	X	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>
A11'	-	-	<ul> <li>Image: A second s</li></ul>	-	-	-	<ul> <li>Image: A second s</li></ul>	-	1	<ul> <li>Image: A second s</li></ul>	X	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>
A14	-	-	<ul> <li>Image: A second s</li></ul>	-	-	-	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	×	-
A15	<ul> <li>Image: A set of the set of the</li></ul>	-	-	-	-	-	-	-	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A set of the set of the</li></ul>	-	X

## Results

	A0	A1	A2'	A3	A4	A5	A7'	A8	A9'	A10'	A11'	A14	A15
A0													
A1	1	×	<ul> <li>Image: A second s</li></ul>	-	-	-	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	-
A2'	-	-	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>
A3	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>
A4	1	-	<ul> <li>Image: A second s</li></ul>	-	×	-	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	-
A5	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>
A7'	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>
A8	1	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>
A9'	-	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>
A10'	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>
A11'	1	-	<ul> <li>Image: A second s</li></ul>	-	-	-	<ul> <li>Image: A second s</li></ul>	-	1	<ul> <li>Image: A second s</li></ul>	X	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>
A14	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A second s</li></ul>
A15	-	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	X

## Results

	A0	A1	A2'	A3	A4	A5	A7'	A8	A9'	A10'	A11'	A14	A15
A0													
A1	1	×	<ul> <li>Image: A second s</li></ul>	-	-	-	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>
A2'	-	<ul> <li>Image: A set of the set of the</li></ul>	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>
A3	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>
A4	1	-	<ul> <li>Image: A second s</li></ul>	-	×	-	<ul> <li>Image: A set of the set of the</li></ul>	-	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>
A5	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>
A7'	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>
A8	1	-	<ul> <li>Image: A second s</li></ul>	-	-	-	-	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>
A9'	-	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>
A10'	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>
A11'													
A14	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A second s</li></ul>
A15	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	X
	A0	A1	A2'	A3	A4	A5	A7'	A8	A9'	A10'	A11'	A14	A15
------	----	--	--	--	--	--	--	--	--	--	--	--	--
A0													
A1	1	×	<ul> <li>Image: A second s</li></ul>	-	-	-	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>
A2'	-	-	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>
A3	-	-	<ul> <li>Image: A second s</li></ul>	X	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A start of the start of</li></ul>
A4	1	-	<ul> <li>Image: A second s</li></ul>	-	×	-	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>
A5	-	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>
A7'													
A8	1	-	<ul> <li>Image: A second s</li></ul>	-	-	-	<ul> <li>Image: A second s</li></ul>	×	1	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>
A9'	-	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>
A10'	-	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>
A11'													
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A15	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	1	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	X

	A0	A1	A2'	A3	A4	A5	A7'	A8	A9'	A10'	A11'	A14	A15
A0													
A1	-	×	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	1		1	<ul> <li>Image: A second s</li></ul>
A2'	<ul> <li>Image: A second s</li></ul>	-	×	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>		<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>	-	<ul> <li>Image: A second s</li></ul>
A3	-	-	<ul> <li>Image: A set of the set of the</li></ul>	X	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A start of the start of</li></ul>
A4	-	-	<ul> <li>Image: A set of the set of the</li></ul>	-	×	-	-	-	<ul> <li>Image: A second s</li></ul>	1	1	1	<ul> <li>Image: A second s</li></ul>
A5	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A second s</li></ul>		<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A set of the set of the</li></ul>		-	<ul> <li>Image: A second s</li></ul>
A7'													
A8	-	-	<ul> <li>Image: A second s</li></ul>	-	-	-	<ul> <li>Image: A second s</li></ul>	×	<ul> <li>Image: A second s</li></ul>	1	1	<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A second s</li></ul>
A9'	-	-	<ul> <li>Image: A second s</li></ul>	-	-	-	<ul> <li>Image: A second s</li></ul>	-	×	<ul> <li>Image: A second s</li></ul>		<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>
A10'	-	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	×		<ul> <li>Image: A set of the set of the</li></ul>	<ul> <li>Image: A start of the start of</li></ul>
A11'													
A14	-	-	<ul> <li>Image: A second s</li></ul>	-	-	-	<ul> <li>Image: A second s</li></ul>	-	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>		×	<ul> <li>Image: A second s</li></ul>
A15	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	<ul> <li>Image: A second s</li></ul>	1	1	1	<ul> <li>Image: A second s</li></ul>	X



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#### Thank you!