Improving Angular Speed Uniformity by Piecewise Radical Reparameterization

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Outline

Problem

Method

Algorithm & Example

Summary

Intuition: How to Drive through A Corner?

Keep the angular speed as a constant!

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Keep the angular speed as a constant!

Angular Speed Uniformity

- Consider a parameterization $p = (x, y) : [0, 1] \to \mathbb{R}^2$

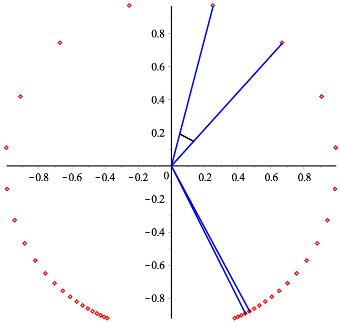
$$\omega_p = |\theta_p'| = \frac{|x'y'' - x''y'|}{x'^2 + y'^2}, \quad \mu_p = \int_0^1 \omega_p \, dt, \quad \sigma_p^2 = \int_0^1 (\omega_p(t) - \mu_p)^2 \, dt$$

- Angular Speed Uniformity

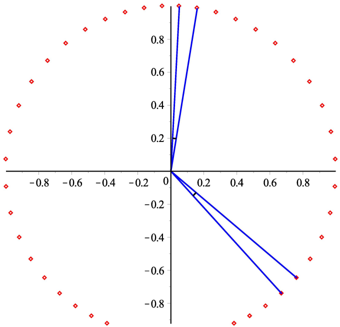
$$u_p = \begin{cases} \dfrac{1}{1 + \sigma_p^2/\mu_p^2}, & \text{if } \mu_p \neq 0; \\ 1, & \text{if } \mu_p = 0. \end{cases}$$

- $u_p \in (0, 1]$; $u_p = 1 \Leftrightarrow \omega_p =$ constant
- u_p can measure the goodness of p.

Examples: Angular Speed Uniformity



$u_{p_1} \doteq 0.482$, "bad"



$u_{p_2} \doteq 0.977$, "good"

Arc-angle Parameterization

- When $u_p = 1$, p is called an *arc-angle parameterization*.
  e.g. $p = (\cos t, \sin t)$ is an arc-angle parameterization.
- How to compute the arc-angle reparameterization p^* of p?
  Answer. Proper parameter transformation r_p: $p^* = p \circ r_p$
- $r : [0, 1] \to [0, 1]$ is called a *proper parameter transformation* if
  C1. $r(0) = 0$, $r(1) = 1$;
  C2. for all $s \in [0, 1]$, $r'(s) \geq 0$.

Arc-angle Reparameterization

Problem A

Given: $p \in \mathbb{Q}(t)^2$.

Find: a proper parameter transformation $r_p(s)$ such that $p \circ r_p$ is an arc-angle parameterization.

Theorem

Let

$$\psi_p(t) = \frac{1}{\mu_p} \int_0^t \omega_p(t)\, dt,$$

and $r_p = \psi_p^{-1}$. Then $u_{p \circ r_p} = 1$, i.e. $p \circ r_p$ is an arc-angle reparameterization of p.

- r_p is called a *uniformizing parameter transformation*.

Rational Approximation of Arc-angle Reparameterization

Q: Is arc-angle parameterization rational?

A: The answer is "No" except for straight lines.

Problem B

Given: $p \in \mathbb{Q}(t)^2$.

Find: a rational p^* such that $u_{p^*} \doteq 1$ or equivalently a rational r such that $u_{p\circ r} \doteq 1$.

Rational Approximation of Arc-angle Reparameterization

- Approach: piecewise rational functions of low degree

  e.g. piecewise Möbius transformation

- Related Work:

  Patterson & Bajaj '89, Kosters '91, Yang et al. '12 & '13

- Drawback: only valid for curves without inflation points

  i.e. $\omega_p(t) \neq 0$ for $\forall t \in [0, 1]$

Difficulty

- $u_{por} \doteq 1 \Rightarrow \omega_{por} \doteq \mu_{por} = \mu_p$
- $\omega_{por} = (\omega_p \circ r) \cdot r'$
- If $\omega_p(\bar{t}) = 0$, $\omega_p \circ r(\bar{s}) = 0 \wedge \omega_{por}(\bar{s}) \neq 0 \Rightarrow r'(\bar{s}) = +\infty$
- $r(s)$ $(0 \leq s \leq 1)$ is continuous and rational $\Rightarrow r'(s)$ bounded $\times$
- Alternative Choice: Piecewise Radical Transformation

Notations

- $p'(t) = \left\{ \dfrac{X_1(t)}{W(t)}, \dfrac{X_2(t)}{W(t)} \right\}$ where $\gcd(X_1, X_2, W) = 1$
- $\omega_p = \dfrac{|X_1'(t)X_2(t) - X_2'(t)X_1(t)|}{X_1^2(t) + X_2^2(t)}$
- Multiplicity of t_i in ω_p: $\mu_i =$mult(ω_p, t_i)

$F(t) = X_1'(t)X_2(t) - X_2'(t)X_1(t) = \prod_{i=1}^{k}(t - t_i)^{\mu_i} \cdot \zeta(t)$

Notations

Let $T = (t_0, \ldots, t_N), S = (s_0, \ldots, s_N)$ be such that

- $0 = t_0 < \cdots < t_N = 1$, $0 = s_0 < \cdots < s_N = 1$;
- either $\omega_p(t_i)$ or $\omega'_p(t_i)$ is zero for $0 < i < N$;
- the multiplicity of t_i in ω_p is μ_i;
- $\omega_p(t) \neq 0$ for all $t \in (t_i, t_{i+1})$.

Piecewise Radical Transformation

Definition

We call φ an elementary piecewise radical transformation associated to p if

$$\varphi(s) = \begin{cases} \vdots & \\ \varphi_i(s), & \text{if} \quad s \in [s_i, s_{i+1}]; \\ \vdots & \end{cases}$$

where

$$\varphi_i(s) = \begin{cases} t_i + \Delta t_i \sqrt[\mu_i+1]{\tilde{s}}, & \text{if } \omega_p(t_i) = 0; \\ t_i + \Delta t_i (1 - \sqrt[\mu_{i+1}+1]{1 - \tilde{s}}), & \text{if } \omega_p(t_{i+1}) = 0; \\ t_i + \Delta t_i \cdot \tilde{s}, & \text{otherwise.} \end{cases}$$

and $\Delta t_i = t_{i+1} - t_i$, $\Delta s_i = s_{i+1} - s_i$, $\tilde{s} = (s - s_i)/\Delta s_i$.

Definition

Let $Z = (z_0, \ldots, z_N)$ and $\alpha = (\alpha_0, \ldots, \alpha_{N-1})$ be such that

$$0 = z_0 < \cdots < z_N = 1, \qquad 0 < \alpha_0, \ldots \alpha_{N-1} < 1$$

and p, S be as given before. Then m is called a piecewise Möbius transformation associated to p if m is of the form:

$$m(z) = \begin{cases} \vdots & \\ m_i(z), & \text{if} \quad z \in [z_i, z_{i+1}]; \\ \vdots & \end{cases}$$

where

$$m_i(z) = s_i + \Delta s_i \cdot \frac{(1-\alpha_i)\tilde{z}}{(1-\alpha_i)\tilde{z} + \alpha_i(1-\tilde{z})} \tag{1}$$

and $\Delta z_i = z_{i+1} - z_i$, $\Delta s_i = s_{i+1} - s_i$, $\tilde{z} = (z - z_i)/\Delta z_i$.

Piecewise Radical Reparameterization

Problem C

Given: $p \in \mathbb{Q}(t)^2$.

Find: φ, m over $[0, 1]$ such that

- $u_{p\circ\varphi\circ m} \doteq 1$;
- $\forall z \in [0, 1]$, $\omega_{p\circ\varphi\circ m}(z) \neq 0$.

Outline

Property of $\varphi(s)$ (1)

- Let $t = \varphi_i(s)$ and $\tilde{t} = (t - t_i)/\Delta t_i$. Then

$$\tilde{t} = \begin{cases} \sqrt[\mu_i+1]{\tilde{s}}, & \text{if } \omega_p(t_i) = 0; \\ 1 - \sqrt[\mu_{i+1}+1]{1 - \tilde{s}}, & \text{if } \omega_p(t_{i+1}) = 0; \\ \tilde{s}, & \text{otherwise.} \end{cases}$$

- $\varphi(s)$ with C^0 continuity while $\varphi'(s)$ discontinuous
- $\omega_{p\circ\varphi}(s)$ discontinuous at $s = s_i$

Property of $\varphi(s)$ (2)

- $\omega_{p\circ\varphi}(s) \neq 0$ for $\forall s \in [0, 1]$

Example

Consider $p = (t, t^3)$.

- $\omega_p = \dfrac{6\,t}{9\,t^4 + 1}$
- $t = 0$ is a zero of ω_p with multiplicity 1
- $T = [0, 1]$, $S = [0, 1]$

Construct $\varphi(s) = \sqrt{s}$. It follows that

$$\omega_{p\circ\varphi}(s) = (\omega_p \circ \varphi)(s) \cdot \varphi'(s) = \frac{6\,\sqrt{s}}{9\,s^2 + 1} \cdot \frac{1}{2\,\sqrt{s}} = \frac{3}{9\,s^2 + 1}.$$

Choice of T

Properties of φ^{-1} ($\varphi = \varphi_{T,S,\alpha}$)	Properties of r_p^{-1} ($r_p^{-1} = \int_0^t \omega_p(\gamma)\, d\gamma / \mu_p$)
$\varphi^{-1}(0) = 0,\ \varphi^{-1}(1) = 1$	$r_p^{-1}(0) = 0,\ \ r_p^{-1}(1) = 1$
$(\varphi^{-1})'(t) \geq 0$ over $[0, 1]$	$(r_p^{-1})'(t) \geq 0$ over $[0, 1]$
$(\varphi^{-1})'(t)$ is monotonic over $[t_i, t_{i+1}]$	If $\omega_p(t_i)\omega_p'(t_i) = 0$, then $(r_p^{-1})'(t)$ is monotonic over $[t_i, t_{i+1}]$

Conclusion

Choose $T = [0, \ldots, t_i, \ldots, 1]$ s.t. $\omega_p(t_i)\omega_p'(t_i) = 0$

Determination of S

Theorem

The uniformity $u_{p\circ\varphi}$ reaches the maximum when

$$s_i = s_i^* = \frac{\sum_{k=0}^{i-1} \sqrt{L_k}}{\sum_{k=0}^{N-1} \sqrt{L_k}}$$

where

$$L_k = \begin{cases} \Delta t_k \int_{t_k}^{t_{k+1}} \frac{\omega_p^2(t)}{(\mu_k + 1)\tilde{t}^{\mu_k}}\, dt & \text{if} \quad \omega_p(t_k) = 0; \\ \Delta t_k \int_{t_k}^{t_{k+1}} \frac{\omega_p^2(t)}{(\mu_{k+1} + 1)(1 - \tilde{t})^{\mu_{k+1}}}\, dt & \text{if} \quad \omega_p(t_{k+1}) = 0; \\ \Delta t_k \int_{t_k}^{t_{k+1}} \omega_p^2(t)\, dt, & \text{otherwise.} \end{cases}$$

Determination of α and Z

Theorem

Let q be such that $\omega_q(s) \neq 0$ over $[0, 1]$ and m be a piecewise Möbius transformation determined by S, Z and α. For given S, $u_{q \circ m}$ reaches the maximum when

$$\alpha_i = \alpha_i^* = \frac{1}{1 + \sqrt{C_i/A_i}}, \quad z_i = z_i^* = \frac{\sum_{k=0}^{i-1} \sqrt{M_k}}{\sum_{k=0}^{N-1} \sqrt{M_k}}$$

where

$$A_i = \int_{s_i}^{s_{i+1}} \omega_q^2 \cdot (1 - \tilde{s})^2 ds, \qquad B_i = \int_{s_i}^{s_{i+1}} \omega_q^2 \cdot 2\tilde{s}(1 - \tilde{s}) ds,$$

$$C_i = \int_{s_i}^{s_{i+1}} \omega_q^2 \cdot \tilde{s}^2 ds, \qquad M_k = \Delta s_k \left(2\sqrt{A_k C_k} + B_k\right).$$

Express A_i, B_i and C_i via p (1)

$$A_i = \begin{cases} \dfrac{\Delta t_i}{\Delta s_i} \int_{t_i}^{t_{i+1}} \dfrac{\omega_p^2(t)}{(\mu_i + 1)\tilde{t}^{\mu_i}} \cdot (1 - \tilde{t}^{\mu_i+1})^2 dt, & \text{if } \omega(t_i) = 0; \\ \dfrac{\Delta t_i}{\Delta s_i} \int_{t_i}^{t_{i+1}} \dfrac{\omega_p^2(t)}{\mu_{i+1} + 1} \cdot (1 - \tilde{t})^{\mu_{i+1}+2} dt, & \text{if } \omega(t_{i+1}) = 0; \\ \dfrac{\Delta t_i}{\Delta s_i} \int_{t_i}^{t_{i+1}} \omega_p^2(t) \cdot (1 - \tilde{t})^2 dt, & \text{otherwise}; \end{cases}$$

$$B_i = \begin{cases} \dfrac{\Delta t_i}{\Delta s_i} \int_{t_i}^{t_{i+1}} \dfrac{\omega_p^2(t)}{\mu_i + 1} \cdot 2\tilde{t}(1 - \tilde{t}^{\mu_i+1}) dt, & \text{if } \omega(t_i) = 0; \\ \dfrac{\Delta t_i}{\Delta s_i} \int_{t_i}^{t_{i+1}} \dfrac{\omega_p^2(t)}{\mu_{i+1} + 1} \cdot 2[1 - (1 - \tilde{t})^{\mu_{i+1}+1}](1 - \tilde{t}) dt, & \text{if } \omega(t_{i+1}) = 0; \\ \dfrac{\Delta t_i}{\Delta s_i} \int_{t_i}^{t_{i+1}} \omega_p^2(t) \cdot (1 - \tilde{t})^2 dt, & \text{otherwise}; \end{cases}$$

Express A_i, B_i and C_i via p (2)

$$C_i = \begin{cases} \frac{\Delta t_i}{\Delta s_i} \int_{t_i}^{t_{i+1}} \frac{\omega_p^2(t)}{\mu_i + 1} \cdot \tilde{t}^{\mu_i+2} dt, & \text{if } \omega(t_i) = 0; \\ \frac{\Delta t_i}{\Delta s_i} \int_{t_i}^{t_{i+1}} \frac{\omega_p^2(t)}{(\mu_{i+1} + 1)(1-\tilde{t})^{\mu_{i+1}}} \cdot [1 - (1-\tilde{t})^{\mu_{i+1}+1}]^2 dt, & \text{if } \omega(t_{i+1}) = 0; \\ \frac{\Delta t_i}{\Delta s_i} \int_{t_i}^{t_{i+1}} \omega_p^2(t) \cdot (1-\tilde{t})^2 dt, & \text{otherwise.} \end{cases}$$

Challenge: Numerical Instability (appearing in L_i too)

$$L_i = \begin{cases} \Delta t_i \int_{t_i}^{t_{i+1}} \frac{\omega_p^2(t)}{(\mu_i + 1)\tilde{t}^{\mu_i}} dt & \text{if } \omega_p(t_i) = 0; \\ \Delta t_i \int_{t_i}^{t_{i+1}} \frac{\omega_p^2(t)}{(\mu_{i+1} + 1)(1-\tilde{t})^{\mu_{i+1}}} dt & \text{if } \omega_p(t_{i+1}) = 0; \\ \Delta t_i \int_{t_i}^{t_{i+1}} \omega_p^2(t)\, dt, & \text{otherwise.} \end{cases}$$

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Instability Solution

- $t = \gamma$ is a zero of $\omega_p(t)$ with multiplicity μ_i and $\gamma \doteq t_i$
- $\omega_p^2(t) = G/H$ and $\mathrm{mult}(G, \gamma) \geq \mu_i$
- $G(t) = (t-\gamma)^{m_i} Q(t,\gamma) + R(t,\gamma) \Rightarrow R(t,\gamma) \equiv 0$

$$
\begin{aligned}
L_i &= \Delta t_i^{\mu_i+1} \int_{t_i}^{t_{i+1}} \frac{\omega_p^2(t)}{(\mu_i+1)\,(t-\gamma)^{\mu_i}}\, dt \\
&\doteq \frac{\Delta t_i^{\mu_i+1}}{\mu_i+1} \cdot \int_{t_i}^{t_{i+1}} \frac{Q(t,t_i)}{H(t)}\, dt. \\
L_{i-1} &\doteq \frac{(-1)^{\mu_i}\,\Delta t_{i-1}^{\mu_i+1}}{\mu_i+1} \cdot \int_{t_{i-1}}^{t_i} \frac{Q(t,t_i)}{H(t)}\, dt.
\end{aligned}
$$

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Algorithm: Piecewise Radical Reparameterization

Input: p

Output: r, a piecewise radical transformation of p such that $u_{p\circ r} > u_p$

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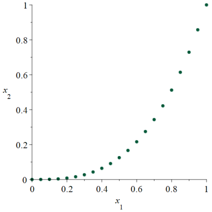
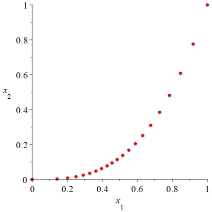
1. Compute T by solving $\omega_p(t)\omega_p'(t) = 0$.
2. Compute S, Z and α which globally minimize $u_{p\circ\varphi}$.
3. Construct φ with T, S and m with S, Z, α.
4. $r \leftarrow \varphi \circ m$.
5. Return r.

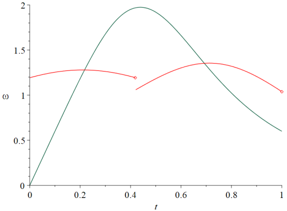
Example: Cubic Curve $p = (t, t^3)$



Original param



Radical reparam



Angular speed

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Summary

- Propose the concept of piecewise radical transformation
- Prove $\omega_{p\circ\varphi}(s) \neq 0$ for $\forall s \in [0, 1]$
- Design an algorithm for computing a radical reparameterization with uniformity close to 1

Future Work

- Search for other efficient parameter transformations to make ω_{p^*} continuous
- Generalize the framework for improving the angular speed uniformity of parametric curves to surfaces or real varieties of higher dimension

Thank you!